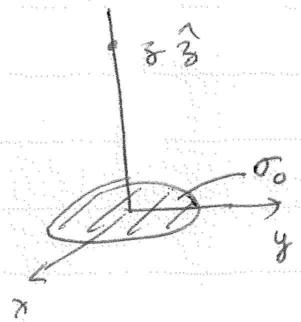
Ex: potential from a charged disc



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int da' \, \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

here $da = r' \, d\phi \, dr'$ where r' is cylindrical radial coordinate

$\sigma(\vec{r}') = \sigma_0$

$\vec{r} = z\hat{z}$

$\vec{r}' = r'\hat{r}$

$$|\vec{r}-\vec{r}'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}')^{\frac{1}{2}}$$

$$= (z^2 + r'^2)^{1/2}$$

$$V(z\hat{z}) = \frac{1}{4\pi\varepsilon_0}\int_0^R dr' \int_0^{2\pi} d\phi \, \frac{r'\sigma_0}{(z^2+r'^2)^{1/2}}$$

assume $z > 0$

$$= \frac{2\pi\sigma_0}{4\pi\varepsilon_0}\int_0^R dr' \, \frac{r'}{(z^2+r'^2)^{1/2}}$$

$$= \frac{\sigma_0}{2\varepsilon_0}\left[(z^2+r'^2)^{1/2}\right]_0^R$$

$$V(z\hat{z}) = \frac{\sigma_0}{2\varepsilon_0}\left[\sqrt{z^2+R^2} - z\right]$$

We can now compute $\vec{E}_z(z\hat{z}) = -\frac{\partial V}{\partial z}$

$$E_z(z) = -\frac{\sigma_0}{2\varepsilon_0}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]$$

same answer as found in earlier HW.

Note: we cant directly compute E_x and E_y as we only

~~to~~ know V ~~on~~ the z-axis – so we can't compute $\frac{\partial V}{\partial x}$ or $\frac{\partial V}{\partial y}$.

Often it is easier to compute $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

and take $\vec{E} = -\vec{\nabla}V$, rather than directly compute

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \rho(\vec{r}\,') \, \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

as ~~the~~ the integral for $V(\vec{r})$ is a <u>scalar</u> integral while the integral for $E(\vec{r})$ is a <u>vector</u> integral.

Electrostatic potential from integrating $\vec{E}(\vec{r})$

Examples : spherical shell of radius R with uniform σ on surface

$$\vec{E}(\vec{r}) = E(r)\hat{r} = \begin{cases} 0 & r < R \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r \geq R \end{cases} \qquad Q = 4\pi R^2 \sigma$$

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

can choose any ref point $\vec{r}_0$

any path from $\vec{r}_0$ to $\vec{r}$

So that $V(\vec{r}) \to 0$ as $r \to \infty$ we choose $\vec{r}_0 = \infty$ and take path to be in radial $\hat{r}$ direction

$$V(\vec{r}) = -\int_{\infty}^{r} d\vec{l}' \cdot \vec{E}(\vec{r}') = \int_{r}^{\infty} dr' \, \hat{r} \cdot \hat{r} E(r')$$

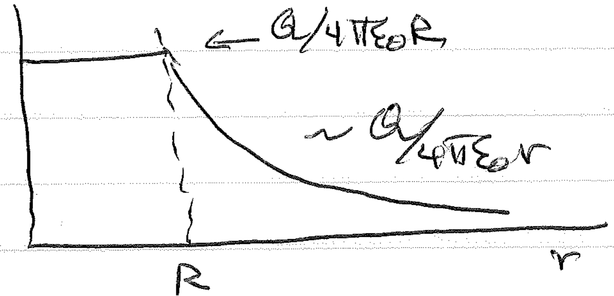
$$= \int_{r}^{\infty} dr' \frac{Q}{4\pi\epsilon_0 r'^2} = \left[\frac{-Q}{4\pi\epsilon_0 r'}\right]_{r}^{\infty} = \boxed{\frac{Q}{4\pi\epsilon_0 r} \quad r \geq R}$$

for $r < R$

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}') = -\int_{\infty}^{R} dr' \, E(r') - \int_{R}^{r} dr' \, E(r')$$

$$= V(R) - 0 = \boxed{\frac{Q}{4\pi\epsilon_0 R} = V(\vec{r}) \quad r < R}$$

$V(r) =$



Although $\vec{E}$ is discontinuous at $r = R$, potential V is continuous

Another way to do the integral $-\int_{\infty} d\vec{l}' \cdot \vec{E}(\vec{r}')$

let us parameterize the path from ∞ to $\vec{r}$ as

$\vec{r}'(t) = \frac{r}{t}\hat{r}$ as t varies from 0 to 1, the curve $\vec{r}'(t)$ sweeps out the points on the radial path from ∞ to $\vec{r}$

t is just a parameter
it is NOT time

We can then write (see earlier discussion of line integrals

$$V(r) = -\int_{\infty}^{r} d\vec{l}' \cdot \vec{E}(\vec{r}') = -\int_{0}^{1} dt \, \frac{d\vec{r}'}{dt} \cdot \vec{E}(\vec{r}'(t))$$

$$= -\int_0^1 dt \left(\frac{-r}{t^2}\hat{r}\right) \cdot \hat{r} \, \frac{Q}{4\pi\varepsilon_0 (r/t)^2}$$

$$= +\int_0^1 dt \, \frac{Q}{4\pi\varepsilon_0} \frac{1}{r} = \frac{Q}{4\pi\varepsilon_0 r} \int_0^1 dt = \frac{Q}{4\pi\varepsilon_0 r}$$

Same answer as before $V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r}$ for $r > R$

For sphere of uniform sphere radius R uniform charge density ρ

$$\vec{E}(\vec{r}) = E(r)\hat{r} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{R^3}\hat{r} & r<R \\[2ex] \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & r>R \end{cases}$$

$$V(\vec{r}) = -\int_{\infty}^{r} d\vec{l}'\cdot\vec{E}(\vec{r}')$$

For $r>R$ the calculation is the same as for the spherical shell and $\boxed{V(r) = \dfrac{Q}{4\pi\varepsilon_0 r} \quad r>R}$

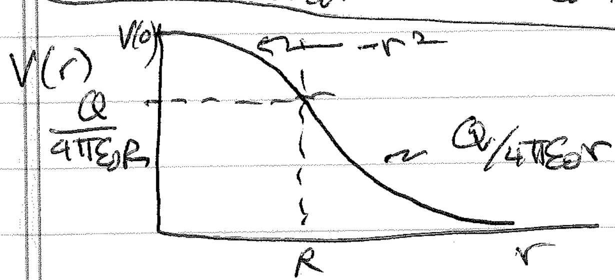
For $r<R$

$$V(\vec{r}) = -\int_{\infty}^{R} d\vec{l}'\cdot E(\vec{r}') - \int_{R}^{r} d\vec{l}'\cdot E(\vec{r}') = V(R) - \int_{R}^{r} d\vec{l}\cdot\vec{E}(r)$$

$$= \frac{Q}{4\pi\varepsilon_0 R} - \int_{R}^{r} dr'\,\hat{r}\cdot\hat{r}\,\frac{Q}{4\pi\varepsilon_0 R^3}r'$$

$$= \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 R^3}\int_{R}^{r} dr'\,r' = \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 R^3}\left[\tfrac{1}{2}r^2 - \tfrac{1}{2}R^2\right]$$

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 R} + \frac{1}{2}\frac{Q}{4\pi\varepsilon_0 R} - \frac{Q r^2}{2\,4\pi\varepsilon_0 R^3} \qquad r<R}$$



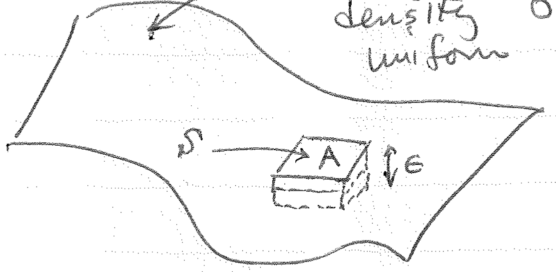
$$V(0) = \frac{Q}{4\pi\varepsilon_0 R}\,\frac{3}{2}$$

Boundary condition at charged surface

$\vec{E}$ is discontinuous at a charged surface —
we saw this w/ charged shell + charged plane problems.

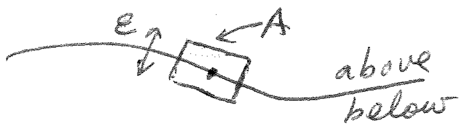
General result

arbitrary charged surface with surface charge density $\sigma(\vec{r})$ where σ need <u>not</u> be uniform



Construct small pillbox with area A, thickness ϵ, which pierces surface at pt $\vec{r}$

side view



Apply Gauss's Law. for small A, small ϵ

$$\int_S d\vec{a}\cdot\vec{E} \cong \underbrace{\left[\vec{E}(\vec{r}_{above})\cdot\hat{n}_{above}\right]A}_{\text{from top}} + \underbrace{\left[\vec{E}(\vec{r}_{below})\cdot\hat{n}_{below}\right]A}_{\text{from bottom}}$$

$$+ \int_{sides} d\vec{a}\cdot\vec{E} = \frac{Q^{encl}}{\varepsilon_0} = \frac{\sigma(\vec{r})A}{\varepsilon_0}$$

let A be fixed as $\epsilon \to 0$. $\Rightarrow \int_{sides} d\vec{a}\cdot\vec{E} = 0$ as thickness of side walls $\to 0$.

$$\Rightarrow \vec{E}(\vec{r}_{above}) \cdot \hat{n}_{above} + \vec{E}(\vec{r}_{below}) \cdot \hat{n}_{below} = \frac{\sigma(\vec{r})}{\varepsilon_0}$$

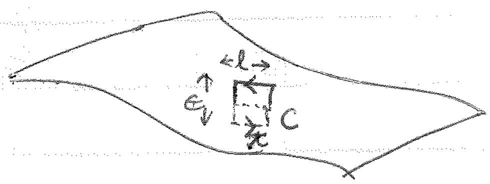
pt at $\vec{r}$ just above charged surface — pt at $\vec{r}$ just below charged surface

But $\hat{n}_{above} = -\hat{n}_{below}$

$$\Rightarrow \left(\vec{E}_{above} - \vec{E}_{below}\right) \cdot \hat{n} = \frac{\sigma(\vec{r})}{\varepsilon_0}$$

So normal component of $\vec{E}$ is discontinuous.

For tangent component: take loop C + use Stokes theorem



$$\oint_C d\vec{l} \cdot \vec{E} = l\left(\vec{E}_{below} \cdot \hat{t} - \vec{E}_{above} \cdot \hat{t}\right)$$

$$= \int d\vec{a} \cdot (\vec{\nabla} \times \vec{E}) = 0$$

$\hat{t}$ is direction of tangent to bottom of curve

$$\Rightarrow (\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{t} = 0$$

for any $\hat{t}$ in plane S

$\rightarrow$ tangential component of $\vec{E}$ is continuous

$$\Rightarrow \boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma(\vec{r})}{\varepsilon_0} \hat{n}}$$

or

$$\boxed{\underbrace{\frac{\partial V_{above}}{\partial n}}_{= -\vec{E}_{above} \cdot \hat{n}} - \underbrace{\frac{\partial V_{below}}{\partial n}}_{= -\vec{E}_{below} \cdot \hat{n}} = -\frac{\sigma(\vec{r})}{\varepsilon_0}}$$

Normal derivative of V is discontinuous, but V is continuous

$$V(\vec{r}_{above}) - V(\vec{r}_{below}) = -\int_{\vec{r}_{below}}^{\vec{r}_{above}} d\vec{l} \cdot \vec{E} \rightarrow 0$$

since $d\vec{l} \rightarrow 0$ as $\vec{r}_{above}$ and $\vec{r}_{below}$ are infinitesimally above + below the surface

Work stored in electrostatic configuration

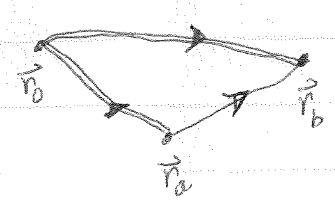
work done on charge Q to move it in an electric field from $\vec{r}_a$ to $\vec{r}_b$

$$W = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{l}$$

$\vec{F}$ is force we must exert on Q to counter the electrostatic force $Q\vec{E}$

$\Rightarrow \quad \vec{F} = -Q\vec{E}(\vec{r})$

$$W = -Q \int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r}) \cdot d\vec{l} = -Q \int_{\vec{r}_a}^{\vec{r}_0} \vec{E} \cdot d\vec{l} - Q \int_{\vec{r}_0}^{\vec{r}_b} \vec{E} \cdot d\vec{l}$$



can always choose path that passes through reference pt $\vec{r}_0$, since $\int \vec{E} \cdot d\vec{l}$ is independent of the path

$$W = Q \int_{\vec{r}_0}^{\vec{r}_a} \vec{E} \cdot d\vec{l} - Q \int_{\vec{r}_0}^{\vec{r}_b} \vec{E} \cdot d\vec{l}$$

since $V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l}$

$$= Q\left[-V(\vec{r}_a)\right] + Q\,V(\vec{r}_b)$$

$$\boxed{W = Q\left[V(\vec{r}_b) - V(\vec{r}_a)\right]}$$

$\Rightarrow$ V is potential energy per unit charge.

To bring Q to point $\vec{r}$, moving it in from ∞ costs work

$$W = Q\left[V(\vec{r}) - V(\infty)\right] = QV(\vec{r})$$

where we choose $V(\infty) = 0$ by convention (ie fixes the arbitrary constant in $V(\vec{r})$)

To build up distribution of point charges q_i at $\vec{r}_i$.

① put q_1 at $\vec{r}_1$: cost no work as there is as yet no $\vec{E}$-field

② move q_2 in from $\vec{r}_0 = \infty$ to position $\vec{r}_2$

$$W_2 = q_2 \, V_1(\vec{r}_2)$$

↑ potential due to charge q_1

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho_1(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\rho_1(\vec{r}\,') = q_1 \delta^3(\vec{r}\,'-\vec{r}_1)$ is charge distr of point charge q_1

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{q_1 \delta^3(\vec{r}\,'-\vec{r}_1)}{|\vec{r}-\vec{r}\,'|}$$

$$\boxed{V_1(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_1|}}$$

potential from a point charge q_1 at position $\vec{r}_1$

$$W_2 = \frac{q_1 q_2}{4\pi\varepsilon_0 |\vec{r}_2-\vec{r}_1|}$$

③ move in q_3 from $\vec{r}_0 = \infty$ to position $\vec{r}_3$

$$W_3 = q_3 \left[V_1(\vec{r}_3) + V_2(\vec{r}_3) \right]$$

pot from q_1 pot from q_2

by superposition, potential from $q_1 + q_2$ is sum of $V_1 + V_2$

$$W_3 = \frac{q_1 q_3}{4\pi\varepsilon_0 |\vec{r}_3 - \vec{r}_1|} + \frac{q_3 q_2}{4\pi\varepsilon_0 |\vec{r}_3 - \vec{r}_2|}$$

(4) move in q_4 from ∞ to $\vec{r}_4$

$$W_4 = q_4 \left[V_1(\vec{r}_4) + V_2(\vec{r}_4) + V_3(\vec{r}_4) \right]$$

$$= \frac{q_4 q_1}{4\pi\varepsilon_0 |\vec{r}_4 - \vec{r}_1|} + \frac{q_4 q_2}{4\pi\varepsilon_0 |\vec{r}_4 - \vec{r}_2|} + \frac{q_4 q_3}{4\pi\varepsilon_0 |\vec{r}_4 - \vec{r}_3|}$$

(5) So on for other charges
add all terms up to get

$$W^{tot} = \frac{1}{4\pi\varepsilon_0} \sum_{(i,j)} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

sum is over all pairs of charges i, j with $i \neq j$ (sometimes see this written as $i < j$)

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{1}{2}\right) \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

factor $\frac{1}{2}$ because we count all pairs twice when ~~to~~ writing the sum like this.

$$\boxed{W = \frac{1}{8\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$

← does not depend on the order in which charges are assembled

work done to assemble charges
= energy stored in charges