Symmetry Planes

Suppose xy plane at $z=0$ is a symmetry plane of the charge distribution.

$$\rho(x,y,z) = \rho(x,y,-z)$$

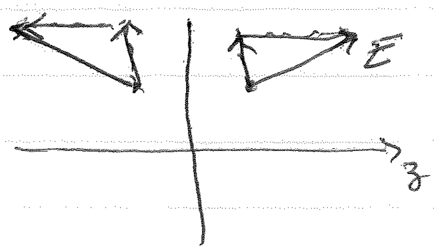
substitute $z \to -z$

$$\rho(x,y,z)/\varepsilon_0 = \rho(x,y,-z)/\varepsilon_0$$

$$\frac{\partial E_x(x,y,z)}{\partial x} + \frac{\partial E_y(x,y,z)}{\partial y} + \frac{\partial E_z(x,y,z)}{\partial z} = \frac{\partial E_x(x,y,-z)}{\partial x} + \frac{\partial E_y(x,y,-z)}{\partial y} + \frac{\partial E_z(x,y,-z)}{\partial(-z)}$$

$$\Rightarrow \left.\begin{aligned} E_x(x,y,z) &= E_x(x,y,-z) \\ E_y(x,y,z) &= E_y(x,y,-z) \end{aligned}\right\} \text{symmetric about } z=0$$

$$E_z(x,y,z) = -E_z(x,y,-z) \quad \} \text{ antisymmetric about } z=0$$



components within planes equal
component $\perp$ plane is equal but opposite

<u>on plane:</u>

$$E_z(x,y,0) = -E_z(x,y,0)$$

$$\Rightarrow E_z(x,y,0) = 0$$

$\Rightarrow$ Component of $\vec{E}$ normal to symmetry plane vanishes.

$\Rightarrow$ $\vec{E}$ at a symmetry plane must lie within the plane.

Denote $\mathbb{R}_z(\vec{r}) = \mathbb{R}_z(x\hat{x} + y\hat{y} + z\hat{z}) \equiv x\hat{x} + y\hat{y} - z\hat{z}$

$$\mathbb{R}_z(\vec{A}) = A_x\hat{x} + A_y\hat{y} - A_z\hat{z}$$

Reflection operator

$$\Rightarrow \quad \vec{E}(\mathbb{R}_z[\vec{r}]) = \mathbb{R}_z[\vec{E}(\vec{r})]$$ if xy plane is symmetry plane

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<u>Antisymmetry planes</u>

Suppose xy plane at $z=0$ is a plane of antisymmetry for the charge distribution

$$\rho(x,y,z) = -\rho(x,y,-z)$$

$\Rightarrow$

$$\frac{\partial E_x(x,y,z)}{\partial x} + \frac{\partial E_y(x,y,z)}{\partial y} + \frac{\partial E_z(x,y,z)}{\partial z} = -\left[\frac{\partial E_x(x,y,-z)}{\partial x} + \frac{\partial E_y(x,y,-z)}{\partial y} + \frac{\partial E_z(x,y,-z)}{\partial(-z)}\right]$$

$$\left.\begin{aligned} E_x(x,y,z) &= -E_x(x,y,-z) \\ E_y(x,y,z) &= -E_y(x,y,-z) \end{aligned}\right\} \text{antisymmetric about } z=0$$

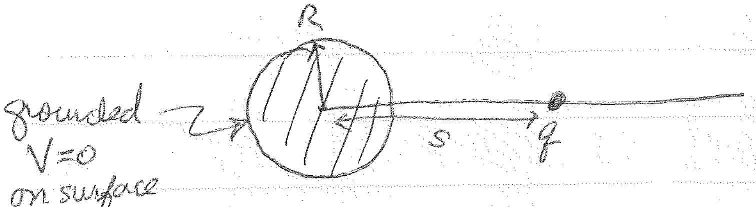
$$\left. E_y(x,y,z) = E_z(x,y,-z) \right\} \text{symmetric about } z=0$$

On plane: $$\left.\begin{aligned} E_x(x,y,0) &= -E_x(x,y,0) \\ E_y(x,y,0) &= -E_y(x,y,0) \end{aligned}\right\} \Rightarrow \begin{aligned} E_x(x,y,0) &= 0 \\ E_y(x,y,0) &= 0 \end{aligned}$$

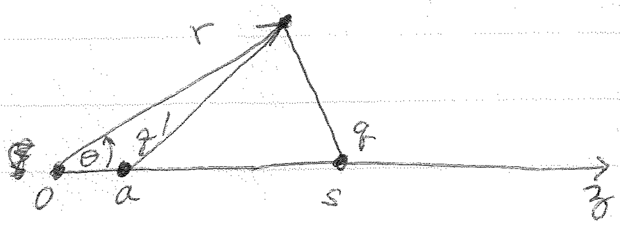
$\Rightarrow$ components of $\vec{E}$ tangential to antisymmetry plane vanish.

$\Rightarrow$ $\vec{E}$ at an antisymmetry plane must lie normal to the plane. (this is why the image charge method for a flat conducting plane works!)

Image charge + spherical conductor.



place an image charge q' inside conductor so that $\vec{E}$ from q and q' is normal to conducting surface.



Prob 13.61

potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{|\vec{r}-s\hat{z}|} + \frac{q'}{|\vec{r}-a\hat{z}|}\right\}$$

$$(\vec{r}-s\hat{z})^2 = r^2+s^2-2s\vec{r}\cdot\hat{z} = r^2+s^2-2sr\cos\theta$$
$$(\vec{r}-a\hat{z})^2 = r^2+s^2-2ar\cos\theta$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2+s^2-2sr\cos\theta}} + \frac{q'}{\sqrt{r^2+a^2-2ar\cos\theta}}\right\}$$

Can we choose q' and a such that $V(R,\theta)^{=0}$ is indep of θ?
ie constant on surface of conductor.

$$V(R,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{(R^2+s^2-2sR\cos\theta)^{1/2}} + \frac{q'}{(R^2+a^2-2aR\cos\theta)^{1/2}}\right\}$$

make denominators look alike

$$R^2+a^2-2aR\cos\theta = \frac{a}{s}\left(\frac{s}{a}R^2 + sa - 2sR\cos\theta\right)$$

if choose $sa = R^2$ ie $\boxed{a = R^2/s}$

then $\frac{sR^2}{a} = s^2$

denominator of 2nd term is $\sqrt{\frac{a}{s}(s^2+R^2-2sR\cos\theta)}$

$\sqrt{a/s} = \sqrt{R^2/s^2} = R/s$

$$V(R,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{R^2+s^2-2sR\cos\theta}} + \frac{q'}{\left(\frac{R}{s}\right)\sqrt{R^2+s^2-2sR\cos\theta}}\right\}$$

choose $\boxed{q' = -q\frac{R}{s}}$ then 2nd term cancells 1st term

So solution to problem is to place image charge $q' = -qR/s$ at position $a\hat{z} = \frac{R^2}{s}\hat{z}$. Nok, since $s > R \Rightarrow a < R$

ie image charge is always inside conductor.

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{|\vec{r}-s\hat{z}|} - \frac{(R/s)}{|\vec{r}-\frac{R^2}{s}\hat{z}|}\right\}$$

$$= \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+s^2-2rs\cos\theta}} - \frac{R/s}{\sqrt{r^2+\frac{R^4}{s^2}-2r\frac{R^2}{s}\cos\theta}}\right\}$$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+s^2-2rs\cos\theta}} - \frac{1}{\sqrt{\frac{s^2r^2}{R^2}+R^2-2rs\cos\theta}}\right\}$$

Surface charge induced on sphere:

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{ \frac{1}{(r^2+s^2-2rs\cos\theta)^{1/2}} - \frac{1}{\left(R^2+\frac{r^2 s^2}{R^2}-2rs\cos\theta\right)^{1/2}} \right\}$$

$$\sigma = \varepsilon_0 \vec{E}\cdot\hat{n}\Big|_{r=R} = \varepsilon_0 E_r\Big|_{r=R} = \varepsilon_0\left(-\frac{\partial V}{\partial r}\right)\Big|_{r=R}$$

$$= -\frac{q}{4\pi}\left\{ \frac{(-\frac{1}{2})(2R-2s\cos\theta)}{(R^2+s^2-2Rs\cos\theta)^{3/2}} - \frac{(-\frac{1}{2})(\frac{2s^2}{R}-2s\cos\theta)}{(R^2+s^2-2Rs\cos\theta)^{3/2}} \right\}$$

$$= \frac{q}{4\pi}\left\{ \frac{R-s^2/R}{(R^2+s^2-2Rs\cos\theta)^{3/2}} \right\} = \frac{q}{4\pi}\frac{s^2}{R}\frac{(R^2/s^2-1)}{s^3\left(1+\frac{R^2}{s^2}-2\frac{R}{s}\cos\theta\right)^{3/2}}$$

$$\sigma(\theta) = \frac{-q}{4\pi}\frac{1}{Rs}\frac{1-(R/s)^2}{\left(1+(R/s)^2-2(R/s)\cos\theta\right)^{3/2}}$$ note $|\sigma(\theta)|$ is largest when $\theta=0$

Note: as charge gets far away, $R/s \to 0$, $\sigma(\theta) \simeq \frac{-q}{4\pi Rs} = \frac{-q}{4\pi R^2}\frac{R}{s}$

$\sigma(\theta) = \frac{q'}{4\pi R^2}$ uniform surface density of total charge $q' = -\frac{R}{s}q$

In general: The total induced charge is $Q_{ind} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, R^2\, \sigma(\theta)$

let $\mu = -\cos\theta$

$$Q_{ind} = 2\pi\int_{-1}^{1} d\mu\, R^2\sigma(\theta) = -\frac{2\pi R^2 q}{4\pi Rs}\left[1-(R/s)^2\right]\int_{-1}^{1} d\mu\left(1+\left(\frac{R}{s}\right)^2+2\frac{R}{s}\mu\right)^{-3/2}$$

$$Q_{ind} = \frac{-qR}{2s}\left[1-(R/s)^2\right]\left[\frac{-\left(1+\left(\frac{R}{s}\right)^2+2\frac{R}{s}\mu\right)^{-1/2}}{(R/s)}\right]_{-1}^{1}$$

$$= +\frac{q}{2}\left\{1-(R/s)^2\right\}\left\{\frac{1}{\sqrt{1+\left(\frac{R}{s}\right)^2+\frac{2R}{s}}} - \frac{1}{\sqrt{1+(R/s)^2-2R/s}}\right\}$$

$$= \frac{q}{2}\left\{1-(R/s)^2\right\}\left\{\frac{1}{1+R/s} - \frac{1}{1-R/s}\right\} = \frac{q}{2}\left[1-(R/s)^2\right]\left[\frac{-2R/s}{1-(R/s)^2}\right]$$

$$\boxed{Q_{ind} = -q\frac{R}{s} = q'}$$

Force of attraction is radial outwards
Force on q is due to electric field of image charge q'

$$\vec{F} = \frac{q q' \hat{z}}{4\pi\varepsilon_0 (s-a)^2} = \frac{-q^2 (R/s) \hat{z}}{4\pi\varepsilon_0 \left(s - \frac{R^2}{s}\right)^2}$$

$$= \frac{-q^2 R s}{4\pi\varepsilon_0 (s^2 - R^2)^2} \hat{z}$$

close to surface of sphere, $s \approx R$
assume $s = R + d$ where $d \ll R$

$$\vec{F} = \frac{-q^2 R s \hat{z}}{4\pi\varepsilon_0 (s-R)^2 (s+R)^2} = \frac{-q^2 R (R+d) \hat{z}}{4\pi\varepsilon_0 (d)^2 (2R+d)^2}$$

$$\simeq \frac{-q^2 R R \hat{z}}{4\pi\varepsilon_0 d^2 4R^2} = \frac{-q^2 \hat{z}}{16\pi\varepsilon_0 d^2}$$

same result as found for the infinite flat plane: when q is so close to sphere that $d \ll R$, it doesnt see curvature of the surface - looks like flat plane.

far from surface of sphere $s \gg R$

$$\vec{F} \simeq \frac{-q^2 R \hat{z}}{4\pi\varepsilon_0 s^3} \sim \frac{1}{s^3}$$ very different from flat plane
very different from pt charge

For a neutral conducting sphere:

If grounded sphere $V=0$, we saw that a net $q' = -q\frac{R}{s}$ was induced.

To handle case of a neutral conducting sphere (not grounded) we need to put back $-q' = +q\frac{R}{s}$ onto sphere. The way to do this, so as to keep $\vec{E}=0$ inside, and to keep $\vec{E} \perp$ surface, is to put it uniformly over surface of sphere. Then $\vec{E}$-field outside the conducting sphere due to the charges induced on the sphere will look just like the field from

$$-q' = q\frac{R}{s} \quad \text{at } \vec{r}=0$$

$$\text{and} \qquad q' = -q\frac{R}{s} \quad \text{at } \vec{r} = \frac{R^2}{s}\hat{z}$$

Force on q outside is

$$\vec{F} = q\left\{ \frac{q'}{4\pi\epsilon_0 (s-a)^2} + \frac{-q'}{4\pi\epsilon_0 s^2} \right\}\hat{z}$$

$$= \frac{qq'}{4\pi\epsilon_0}\left\{ \frac{s^2 - [s^2+a^2-2as]}{s^2(s-a)^2} \right\}\hat{z}$$

$$= \frac{-q^2}{4\pi\epsilon_0}\frac{R}{s}\left\{ \frac{2s\left(\frac{R^2}{s}\right) - \left(\frac{R^2}{s}\right)^2}{s^2\left(s-\frac{R^2}{s}\right)^2} \right\}\hat{z}$$

$$\vec{F} = -\frac{q^2}{4\pi\varepsilon_0}\,\frac{R}{s}\left\{\frac{2R^2 - \frac{R^4}{s^2}}{(s^2-R^2)^2}\right\}\hat{z}$$

$$\vec{F} = -\frac{q^2}{4\pi\varepsilon_0}\left(\frac{R^3}{s}\right)\frac{2-(R/s)^2}{(s^2-R^2)^2}\,\hat{z}$$

Note: since $\frac{R}{s} < 1$ always, then $\vec{F}$ is always in $-\hat{z}$ direction $\Rightarrow$ attractive

for charge <u>close</u> to sphere, $s - R = d \ll R$, we have

$$\vec{F} \simeq \frac{-q^2}{4\pi\varepsilon_0}\,R^2\,\frac{2-1}{(s-R)^2(s+R)^2} = \frac{-q^2}{4\pi\varepsilon_0}\,\frac{R^2}{(2R)^2 d^2}$$

$$\simeq \frac{-q^2}{16\pi\varepsilon_0 d^2}$$

same result as for grounded sphere. Reason: ~~this~~ force from image q' at $\frac{R^2}{s}$ is $\sim\frac{1}{d^2}$. force from $-q'$ at origin is $\sim\frac{1}{R^2}$

when $d \ll R$, $\frac{1}{d^2} \gg \frac{1}{R^2}$ and so force from $-q'$ at origin is negligible compared to force from q' at $\frac{R^2}{s}$.

for charge far from sphere $\frac{R}{s} >> 1$,

$$\vec{F} \simeq \frac{-q^2}{4\pi\varepsilon_0} \frac{R^3}{s} \frac{2 \hat{z}}{s^4} = \frac{-q^2 R^3}{2\pi\varepsilon_0} \frac{\hat{z}}{s^5} \sim \frac{1}{s^5}$$

force decays much more quickly $(\sim \frac{1}{s^5})$ for <u>neutral</u> sphere than it does for <u>grounded</u> sphere $(\sim \frac{1}{s^3})$

<u>Charged conducting sphere</u>

Suppose conducting sphere now has a fixed net charge of Q on it.

To solve this case, it is same as neutral sphere, except now add Q uniformly distributed over surface. This results in same $\vec{E}$ field as for pt Q at origin.

$\Rightarrow \vec{E}$ field outside $\overbrace{\text{looks like}}^{\text{(due to charges on sphere)}}$

$$-q' = q\frac{R}{s} \quad \text{at } \vec{r} = 0$$

$$Q \quad \text{at } \vec{r} = 0$$

$$q' = -q\frac{R}{s} \quad \text{at } \vec{r} = \frac{R^2}{s}\hat{z}$$

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 s^2}\hat{z} - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{R^3}{s}\right) \frac{2-(R/s)^2}{(s^2-R^2)^2}\hat{z}$$

for large s $\quad \sim \frac{1}{s^2} \quad \sim \frac{1}{s^5} \quad$ for s big enough, will be repulsive

for $s \approx R$ $\quad \sim \frac{1}{R^2} \quad \sim \frac{1}{(s-R)^2} \quad$ for $s-R$ small enough, will always attract

Cross over from attractive to repulsive when

$$\frac{Q}{q} = R^3 s \frac{\left(2-(R/s)^2\right)}{(s^2-R^2)^2} = \left(\frac{R}{s}\right)^3 \frac{2-(R/s)^2}{\left[1-\left(\frac{R}{s}\right)^2\right]^2}$$

let $x \equiv \frac{R}{s} \in (0,1)$

$$\frac{Q}{q} = x^3 \frac{(2-x^2)}{(1-x^2)^2}$$

gives 5th order polynomial in x
in general, no analytical solution

$$(1-x^2)^2 \frac{Q}{q} = x^3(2-x^2) = 2x^3 - x^5$$

$$x^5 - 2x^3 + (x^4 - 2x^2 + 1)\frac{Q}{q} = 0$$

$$x^5 + \left(\frac{Q}{q}\right)x^4 - 2x^3 - \left(2\frac{Q}{q}\right)x^2 + \frac{Q}{q} = 0$$

~~If $\frac{Q}{q} \ll 1$, then keep only terms in $\frac{Q}{q}$~~

~~$\Rightarrow x^4 - 2x^2 + 1 = 0 \quad x = \pm 1$~~

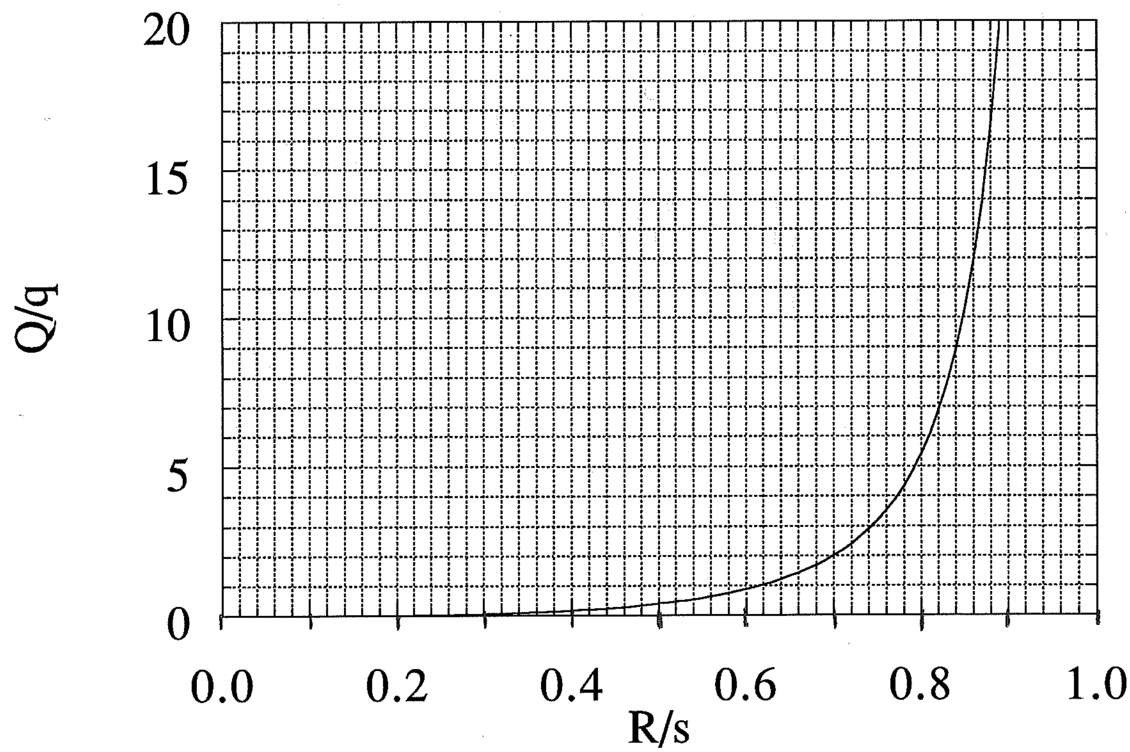
Graphical solution on next pages.

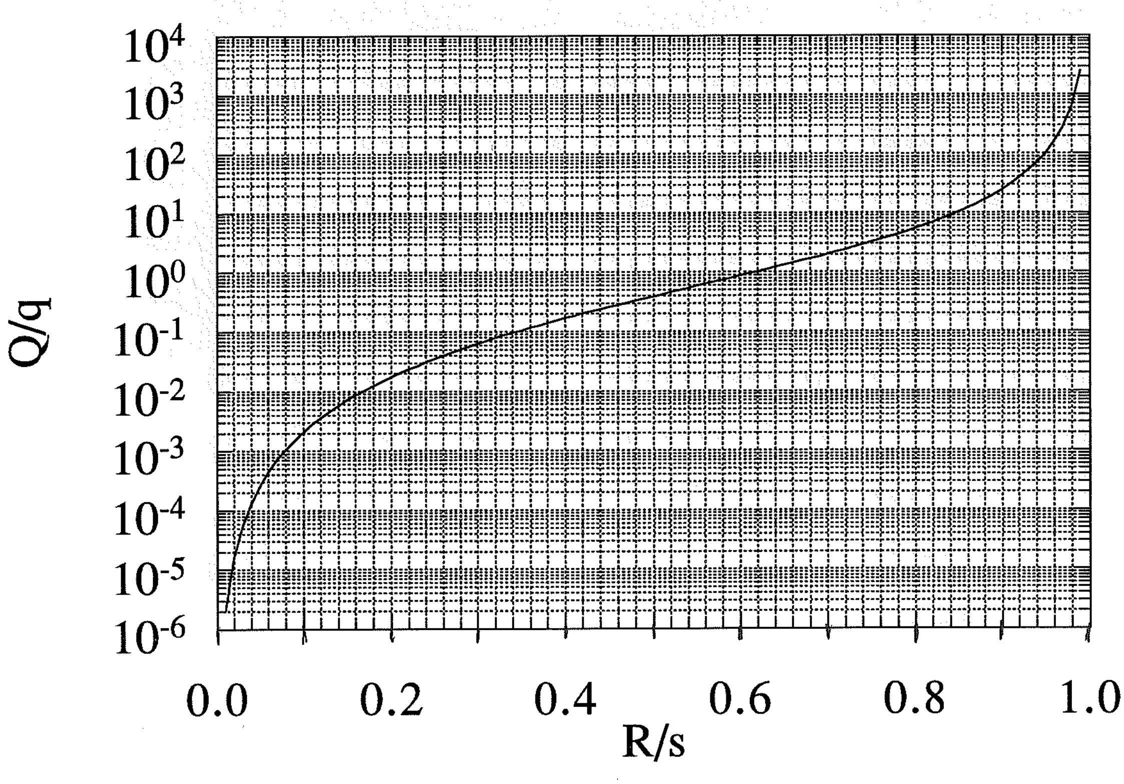
For $\frac{Q}{q} = 1$, cross over is at $\frac{R}{s} \simeq .62$

$$s = \frac{R}{.62} = 1.6R$$

$\frac{Q}{q} = 0.1$, cross over is at $\frac{R}{s} \simeq .36$

$$s \simeq \frac{R}{.36} = 2.8R$$


Q/q
20
15
10
5
0
0.0
0.2
0.4
0.6
0.8
1.0
R/s



Q/q
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.0
0.2
0.4
0.6
0.8
1.0
R/s