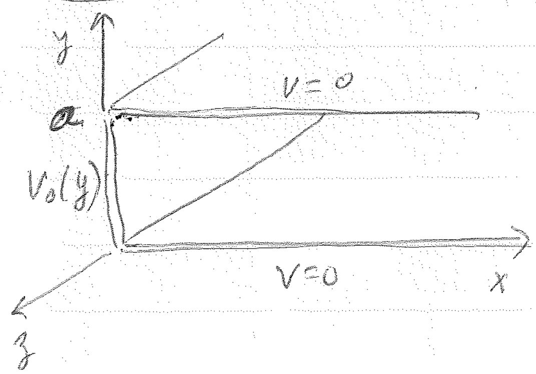
Separation of Variables

Cartesian coords



Find solution to $\nabla^2 V = 0$

with $V(y=0) = 0$

$V(y=a) = 0$

$V(x=0, y, z) = V_0(y)$

$V \to 0$ as $x \to \infty$

everything is indep of z, so $\nabla^2 V$ is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separation of variables: Assume solution of form
$V(x,y) = X(x)\, Y(y)$

$$\Rightarrow \quad Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0 \quad \Rightarrow \quad \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

true for any x or y $\Rightarrow$ $\frac{1}{X}\frac{d^2 X}{dx^2} = C_1$ const

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -C_1$$

Let $C_1 \equiv k^2$ (see why later)

$$\frac{d^2 X}{dx^2} = k^2 X \quad \Rightarrow \quad X(x) = A e^{kx} + B^{-kx}$$

$$\frac{d^2 Y}{dy^2} = -k^2 Y \quad \rightarrow \quad Y(y) = C \sin ky + D \cos ky$$

General solution to $\nabla^2 V = 0$ is

$$V(x, y) = \sum_{k>0} \left(A_k e^{kx} + B_k e^{-kx}\right)\left(C_k \sin ky + D_k \cos ky\right)$$

try to find A_k, B_k, C_k, D_k to satisfy boundary conditions

1) $V \to 0$ as $x \to \infty \Rightarrow A_k = 0$

$V(y=0) = 0$ all $x \Rightarrow D_k = 0$

$V(y=a) = 0$ all $x \Rightarrow \sin ka = 0 \Rightarrow ka = n\pi$, n integer ≥ 0

$$V(x,y) = \sum_{\substack{n=1 \\ \text{integer}}}^{\infty} \underbrace{\cancel{\ }}_{C_n} e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

(n=0 term vanishes)

C_n ~~...~~ $= B_k C_k$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right)$$

$$V(x=0, y) = V_0(y) = \sum_{n=1}^{\infty} \underbrace{C_n}_{\text{Fourier series coefficients of } V_0(y)} \sin\left(\frac{n\pi}{a} y\right)$$

To find C_n in terms of $V_0(y)$ use,

$$\int_0^a dy\, V_0(y) \sin\left(\frac{m\pi}{a} y\right) = \int_0^a dy \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right)$$

Now $$\int_0^a dy\, \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) = \begin{cases} 0 & m \neq n \\ \frac{a}{2} & m = n \end{cases}$$

so all terms in $\sum_n$ vanish except $n = m$ term

$$\int_0^a dy\ V_0(y) \sin\left(\frac{m\pi}{a} y\right) = C_m \frac{a}{2}$$

$$\Rightarrow \boxed{C_n = \frac{2}{a} \int_0^a dy\ V_0(y) \sin\left(\frac{n\pi y}{a}\right)}$$

So solution is $V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right)$

with C_n determined as above

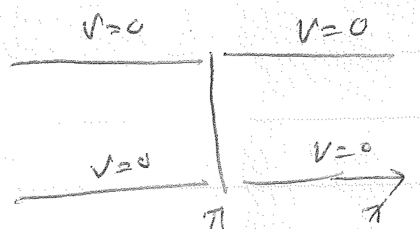
ex: Suppose $V_0(y) = V_0$ const

then $C_n = \frac{2V_0}{a} \int_0^a dy\ \sin\left(\frac{n\pi y}{a}\right)$

$$= \frac{2V_0}{a} \left[-\frac{a}{n\pi} \cos \frac{n\pi y}{a} \right]_0^a$$

$$= \frac{2}{n\pi} V_0 \left[1 - \cos n\pi \right] = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases}$$

prob 3.13



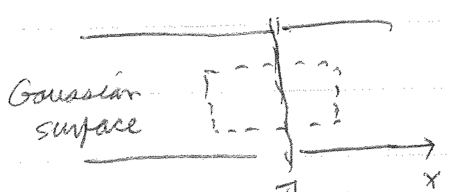
$\sigma(y)$ surface charge on plane at $x=0$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \quad \text{as before for } x>0.$$

$$V(x=0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \quad \text{as before}$$

Only now $V(x=0,y)$ is not a known function $V_0(y)$. What we know is $\sigma(y)$ on the plane. How can we relate this to $V(x=0,y)$? Want to determine the C_n in terms of $\sigma(y)$.

Since plane at $x=0$ is a symmetry plane for charge distrib



$\sigma(y)$

By symmetry $E_x(x=0^+,y) = -E_x(x=0^-,y)$

$$\Rightarrow \underbrace{2E_x(0^+,y)\,da}_{\oint \vec{E}\cdot d\vec{a}} = \underbrace{\frac{\sigma(y)da}{\varepsilon_0}}_{\frac{Q_{enclosed}}{\varepsilon_0}}$$

$$E_x(0^+,y) = \frac{\sigma(y)}{2\varepsilon_0} \Rightarrow -\left.\frac{\partial V}{\partial x}\right|_{x=0^+} = \frac{\sigma(y)}{2\varepsilon_0} \quad \text{as } E_x = -\frac{\partial V}{\partial x}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n\pi}{a} C_n \sin\left(\frac{n\pi y}{a}\right) = \frac{\sigma(y)}{2\varepsilon_0}$$

If we write $\frac{n\pi}{a} C_n \equiv \tilde{C}_n$, then we see that $\tilde{C}_n$ are just the Fourier coefficients of $\sigma(y)/2\varepsilon_0$

$$\Rightarrow \tilde{C}_n = \frac{2}{a}\int_0^a dy\, \frac{\sigma(y)}{2\varepsilon_0} \sin\left(\frac{n\pi y}{a}\right)$$

$$\Rightarrow \boxed{C_n = \frac{a}{n\pi}\tilde{C}_n = \frac{1}{n\pi\varepsilon_0}\int_0^a dy\, \sigma(y) \sin(n\pi y/a)}$$

Solving Poisson's Eqn

Consider spherical shell, radius R, uniform surface charge σ.
Find potential V, by directly solving Poisson's eqn

$$\nabla^2 V = -\rho/\varepsilon_0 \Rightarrow \nabla^2 V = 0 \text{ inside shell}$$
$$= 0 \text{ outside shell}$$

radial symmetry $\Rightarrow$ $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$

$V(\vec{r})$ depends only on $|\vec{r}|$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2 \frac{\partial V}{\partial r} = C_0 \quad \text{constant}$$

$$\frac{\partial V}{\partial r} = \frac{C_0}{r^2}$$

general solution $\Rightarrow$ $\boxed{V(r) = -\frac{C_0}{r} + C_1}$

need to find C_0 and C_1. Apply <u>boundary conditions</u>

<u>inside:</u> We know that V should stay finite inside the shell - it should not diverge there $\Rightarrow C_0 = 0$ inside

<u>outside:</u> We want $V \to 0$ as $r \to \infty$ $\Rightarrow C_1 = 0$ outside

$$\Rightarrow V(r) = \begin{cases} C_1 & \text{inside} \\ -\frac{C_0}{r} & \text{outside} \end{cases}$$

} this agrees with what we know from earlier solution of this problem: V = constant inside. V like pt charge outside

To find C_1 and C_0, need boundary condition at shell $r=R$

1) V is continuous $\Rightarrow V_{in}(R) = V_{out}(R)$

$$\Rightarrow \quad C_1 = -\frac{C_0}{R}$$

2) discontinuity in $\vec{E}$ is $\left(\vec{E}_{out} - \vec{E}_{in}\right)\Big|_{r=R} = \frac{\sigma}{\varepsilon_0}\hat{n}$

$$-\frac{\partial V_{out}}{\partial n}\bigg|_{r=R} + \frac{\partial V_{in}}{\partial n}\bigg|_{r=R} = \frac{\sigma}{\varepsilon_0}$$

for this problem $\hat{n} = \hat{r}$ radial direction

$$\frac{\partial V_{in}}{\partial r} = 0 \quad , \quad \frac{\partial V_{out}}{\partial r}\bigg|_{r=R} = \frac{C_0}{r^2}\bigg|_{r=R} = \frac{C_0}{R^2}$$

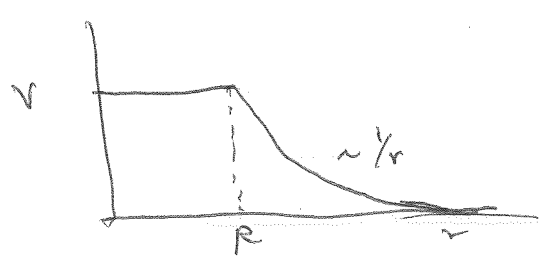
$$\Rightarrow \quad -\frac{C_0}{R^2} + 0 = \frac{\sigma}{\varepsilon_0} \quad \Rightarrow \quad C_0 = -\frac{\sigma R^2}{\varepsilon_0} \quad \Rightarrow C_1 = \frac{\sigma R}{\varepsilon_0}$$

Solution: $$V(r) = \begin{cases} \dfrac{\sigma R}{\varepsilon_0} & \text{inside} \\[2ex] \dfrac{\sigma R^2}{\varepsilon_0 r} & \text{outside} \end{cases}$$

note: total charge on shell is $q = 4\pi R^2 \sigma$, so we could write

$$V(r) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 R} & \text{inside} \\[2ex] \dfrac{q}{4\pi\varepsilon_0 r} & \text{outside} \end{cases}$$

looks like point charge outside



Now we wish to consider cases which do not have spherical symmetry

Spherical coords - Separation of Variables

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^3 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Consider only problems with azimuthal symmetry, so that V is indep of ϕ.

$$\Rightarrow r^2 \nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

assume $V(r,\theta) = R(r)\Theta(\theta)$, plug in, divide by $R\Theta$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}_{= \text{const}} + \underbrace{\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)}_{= -\text{const}} = 0$$

call the const $= \ell(\ell+1)$

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1)$$

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)$$

radial equ: guess solution of form $A r^\alpha$. substitute in

$$\frac{1}{\cancel{A} r^\alpha}\frac{d}{dr}\left(r^2 \alpha A r^{\alpha-1}\right) = \frac{1}{\cancel{A} r^\alpha}\frac{d}{dr}\left(\alpha A r^{\alpha+1}\right)$$

$$= \frac{1}{\cancel{A} r^\alpha}\alpha A(\alpha+1) r^\alpha = \alpha(\alpha+1) = \ell(\ell+1)$$

2nd order differential equ $\Rightarrow$ 2 solutions $\Rightarrow \begin{pmatrix}\alpha = \ell \\ \alpha+1 = \ell+1\end{pmatrix}$ or $\begin{pmatrix}\alpha = -(\ell+1) \\ \alpha+1 = -\ell\end{pmatrix}$

General solution of form $\boxed{R(r) = A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}}$

A_ℓ, B_ℓ arbitrary constants

angular eqn: $\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)(\sin\theta)\Theta$

Solutions are known as the Legendre polynomials $P_\ell(\cos\theta)$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \qquad \text{Rodrigues ~~Fortul~~ Formula}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

In general $P_\ell(x)$ is polynomial of order ℓ, with only
even powers of x if ℓ is even ($\Rightarrow P_\ell(x)$ is symmetric if x even)
odd powers of x if ℓ is odd ($\Rightarrow P_\ell(x)$ is antisymmetric if x odd)

$P_\ell(x=1) = 1$

Note: we only have one solution for each value of ℓ.
But in general, for 2nd order differential eqn, there should be two solutions. Also, we have only discussed solutions for integer $\ell \geq 0$.

However these "2nd solutions" as well as solutions for non integer ℓ, blow up at $\theta = 0$ or $\theta = \pi$, so they are physically unacceptable