General solution is linear combination

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

need to find A_ℓ and B_ℓ so that $V(r,\theta)$ has desired boundary conditions.

$P_\ell(\cos\theta)$ are complete + orthogonal set of functions on interval $[0, \pi]$

orthogonal

$$\Rightarrow \quad \int_{-1}^{1} P_\ell(x) P_m(x)\, dx = \int_0^\pi P_\ell(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta$$

$$= \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{(2m+1)} & \ell = m \end{cases}$$

complete = can write any $V(\theta)$ for $\theta \in [0, \pi]$ as linear combination of the $P_\ell(\cos\theta)$

Ex 6

Potential $V_0(\theta)$ on surface of (empty) sphere of radius R.
Find V inside st $\nabla^2 V = 0$ inside & $V(R,\theta) = V_0(\theta)$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

want solution not to be singular as $r \to 0 \Rightarrow B_\ell = 0$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$$

$$\Rightarrow \int_0^{\pi} d\theta \, \sin\theta \, V_0(\theta) P_m(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^{\pi} d\theta \sin\theta \, P_\ell(\cos\theta) P_m(\cos\theta)$$

$$= A_m R^m \frac{2}{2m+1}$$

$$\Rightarrow \boxed{A_\ell = \frac{(2\ell+1)}{2R^\ell} \int_0^{\pi} d\theta \sin\theta \, V_0(\theta) P_\ell(\cos\theta)}$$

EX 7 : Same as 6, only now find V outside, st $\nabla^2 V = 0$ outside, $V(R,\theta) = V_0(\theta)$

Now we want solution finite as $r \to \infty$. $\Rightarrow A_\ell = 0$

$$V(R,\theta) = \sum_{\ell=0}^{\infty} B_\ell R^{-(\ell+1)} P_\ell(\cos\theta) = V_0(\theta)$$

$$\Rightarrow B_\ell = \frac{2\ell+1}{2R^{-(\ell+1)}} \int_0^{\pi} d\theta \sin\theta \, V_0(\theta) P_\ell(\cos\theta)$$

$$= \frac{2\ell+1}{2} R^{\ell+1} \int_0^{\pi} d\theta \sin\theta \, V_0(\theta) P_\ell(\cos\theta)$$

Prob 3.20

$V_0(\theta)$ on surface given. What is surface charge $\sigma(\theta)$?

outside $V(r,\theta) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

inside $V(r,\theta) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$

$$\Rightarrow \quad V(R,\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$$

$$\Rightarrow \quad \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

$$\int_0^\pi d\theta \sin\theta\, P_m(\cos\theta)\, V_0(\theta) = A_m R^m \frac{2}{2m+1}$$

$$A_m = \frac{2m+1}{2R^m} \int_0^\pi d\theta \sin\theta\, P_m(\cos\theta)\, V_0(\theta)$$

Now

$$\frac{\sigma(\theta)}{\varepsilon_0} = -\frac{\partial V}{\partial r}\Big|_{above} + \frac{\partial V}{\partial r}\Big|_{below} = \sum_\ell \Big[(\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) + \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) \Big]$$

$$= \sum_\ell A_\ell \Big[\ell R^{\ell-1} + \frac{(\ell+1)}{R^{\ell+2}} R^{2\ell+1} \Big] P_\ell(\cos\theta)$$

$$= \sum_\ell A_\ell \big[\ell R^{\ell-1} + (\ell+1) R^{\ell-1} \big] P_\ell(\cos\theta)$$

$$= \sum_\ell A_\ell [2\ell+1] R^{\ell-1} P_\ell(\cos\theta)$$

$$\boxed{\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{\ell=0}^\infty \frac{(2\ell+1)^2}{2R} P_\ell(\cos\theta) \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta)\, V_0(\theta)}$$

Ex 9: Find sol $\nabla^2 V = 0$ inside + outside spherical shell with $\sigma_0(\theta)$ glued on surface

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad r<R$$

$$= \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) \qquad r>R$$

1st boundary condition:

V continuous as cross σ_0, so

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} B_l \frac{1}{R^{l+1}} P_l(\cos\theta)$$

true for all $\theta \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = +A_l R^{2l+1}}$

2nd boundary condition:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\Rightarrow -\left.\frac{\partial V}{\partial r}\right|_{\substack{above \\ r=R}} + \left.\frac{\partial V}{\partial r}\right|_{\substack{below \\ r=R}} = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty} \left\{ B_l \frac{(l+1)}{R^{l+2}} P_l(\cos\theta) + A_l l R^{l-1} P_l(\cos\theta) \right\} = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

substitute for B_l

$$\sum_{l=0}^{\infty} \left(+ A_l (l+1) \frac{R^{2l+1}}{R^{l+2}} + A_l l R^{l-1} \right) P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty} A_\ell R^{\ell-1}\left[\ell+(\ell+1)\right] P_\ell(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

$$+\sum_{\ell=0}^{\infty} A_\ell \underbrace{R^{\ell-1}}_{(2\ell+1)} P_\ell(\cos\theta) = \frac{\sigma_0(\theta)}{\varepsilon_0}$$

$$\Rightarrow \int_0^{\pi} d\theta \sin\theta\, P_m(\cos\theta)\frac{\sigma_0(\theta)}{\varepsilon_0} = A_m R^{m-1}(2m+1)\frac{2}{2m+1}$$

$$\boxed{A_m = \frac{1}{2\varepsilon_0 R^{m-1}}\int_0^{\pi} d\theta \sin\theta\, P_m(\cos\theta)\,\sigma_0(\theta)}$$

Suppose $\sigma_0(\theta) = k\cos\theta = k P_1(\cos\theta)$

then $A_m = 0$ except for $m=1$

$$A_1 = \frac{1}{2\varepsilon_0 R^0}\, k\, \frac{2}{2(1)+1} = \frac{k}{3\varepsilon_0} \Rightarrow B_1 = \frac{k}{3\varepsilon_0}R^3$$

$$V(r,\theta) = \begin{cases} \dfrac{k}{3\varepsilon_0}\, r\cos\theta & r<R \\[2ex] \dfrac{k}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta & r>R \end{cases}$$

$\Leftarrow$ dipolar field

so potential inside sphere is $\frac{k}{3\varepsilon_0} r\cos\theta = \frac{kz}{3\varepsilon_0}$

electric field inside is

$$\vec{E} = -\vec{\nabla}V = -\frac{k}{3\varepsilon_0}\hat{z}$$

field outside sphere with $\sigma = k\cos\theta$.

$$V_{out} = \frac{B_1}{r^2} P_1(\cos\theta) = \frac{kR^3}{3\varepsilon_0 r^2}\cos\theta$$

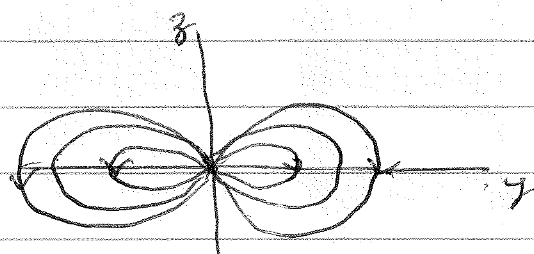
$$\vec{E}_{out} = -\vec{\nabla} V_{out} = -\frac{\partial V_{out}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_{out}}{\partial \theta}\hat{\theta}$$

$$= \frac{2kR^3\cos\theta}{3\varepsilon_0 r^3}\hat{r} + \frac{kR^3}{3\varepsilon_0 r^3}\sin\theta\,\hat{\theta}$$

$$= \frac{kR^3}{3\varepsilon_0 r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

for $\theta = 0, \pi$, $\vec{E}_{out} = \pm\frac{2kR^3}{3\varepsilon_0 r^3}\hat{r}$

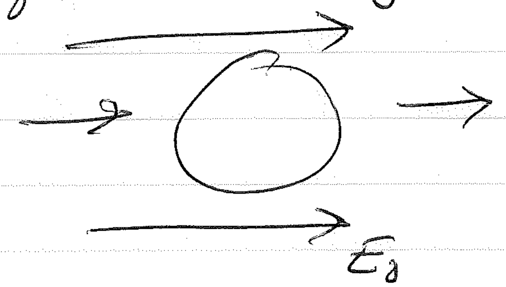
$\theta = \frac{\pi}{2}$ $\vec{E}_{out} = \frac{kR^3}{3\varepsilon_0 r^3}\hat{\theta}$



dipole field

(see prob 3 on HW 2)

This observation allows us to solve a different problem. Conducting sphere in a uniform applied field $\vec{E}_0 = E_0 \hat{z}$.



We know that there must be a $\sigma(\theta)$ induced on the surface of the sphere, that gives rise to an induced field $\vec{E}_{ind}$, such that the total field vanishes inside the conductor

$$\vec{E}_0 + \vec{E}_{ind} = 0$$

If we take $\sigma(\theta) = k \cos\theta$ then we have inside the conducting sphere

$$\vec{E}_0 + \vec{E}_{ind} = \left(E_0 - \frac{k}{3\varepsilon_0}\right)\hat{z}$$

will vanish when $k = 3\varepsilon_0 E_0$

Since this is a solution, it is the only solution!

So a conducting sphere in a uniform applied $E_0\hat{z}$ develops an induced surface charge

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$