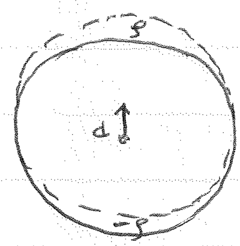
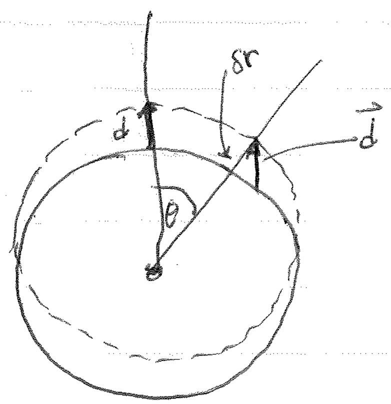
How can we get a surface charge $\sigma(\theta) = k\cos\theta$?

Consider two spheres of equal radii with uniform, but opposite charge densities ρ and $-\rho$. When spheres overlap, net charge densities is zero. Displace spheres by small distance $\vec{d}$.



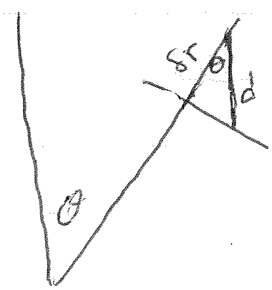
gives $\sigma(\theta)$ with net (+) on top and net (−) on bottom.

This is a uniformly "polarized" sphere: every element of (+) charge in one sphere is displaced by distance $\vec{d}$ from corresponding element of (−) charge in the other sphere



$\sigma(\theta) = \rho\,\delta r$

to find δr



$d\cos\theta = \delta r$

In overlap region found (ex 2.18) $\vec{E} = -\dfrac{\rho d}{3\varepsilon_0}$

$\Rightarrow \sigma(\theta) = \rho d \cos\theta$ $\qquad k = \rho d$

so uniformly polarized sphere has $\sigma(\theta) = \rho d\cos\theta$

$\Rightarrow$ $\vec{E}$ field inside uniformly polarized sphere is $\vec{E} = \dfrac{-\rho\vec{d}}{3\varepsilon_0}$

if $\vec{P} = \rho\vec{d}$ is "polarization density", $\vec{E} = -\dfrac{\vec{P}}{3\varepsilon_0}$

$\vec{E}$ field outside the uniformly polarized sphere
is

$$\vec{E}_{out} = \frac{kR^3}{3\varepsilon_0 r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

$$= \frac{\rho d\, R^3}{3\varepsilon_0 r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

$$= \frac{\mathbb{P}\, R^3}{3\varepsilon_0 r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$ $\vec{\mathbb{P}} = \rho\vec{d}$ is polarization density

or $\vec{P}_{tot} = \mathbb{P}\,\frac{4}{3}\pi R^3$ is total dipole moment

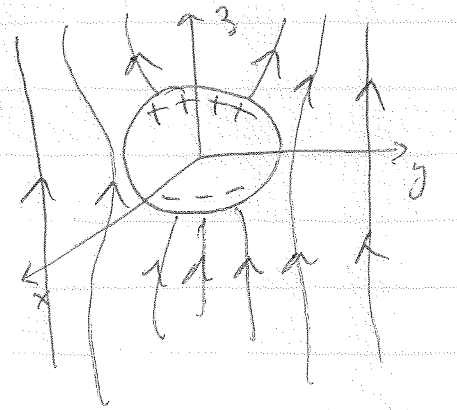
$$\vec{E}_{out} = \frac{\vec{P}_{tot}}{4\pi\varepsilon_0 r^3}\left\{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right\}$$

same field as for a point electric dipole.

(see HW2, prob 3)

Ex 8

Conducting sphere in uniform applied $\vec{E}$ field



$\vec{E}$ induces σ as shown on surface of sphere. σ creates its own $\vec{E}$ field ~~that distorts the applied field $\vec{E}$~~ so that total $\vec{E}$ field is always normal to surface of sphere

By symmetry we see that E_z is symmetric with respect to reflection through z axis.

ie $E_z(x,y,z) = E_z(x,y,-z)$

E_x and E_y are antisymmetric

$$\left.\begin{aligned} E_x(x,y,z) &= -E_x(x,y,-z) \\ E_y(x,y,z) &= -E_y(x,y,-z) \end{aligned}\right\} \Rightarrow \begin{aligned} E_x(x,y,0)&=0 \\ E_y(x,y,0)&=0 \end{aligned}$$

xy plane at $z=0$ is plane of reflection antisymmetry

ie. reflection through z axis ⊕ charge conjugation restores us to original configuration

$\Rightarrow$ xy plane at $z=0$ is equipotential $V=$ constant

(since $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y} = 0$ on the plane)

choose this constant = zero.

$$V(x,y,0) = 0$$

Also surface of sphere is equipotential, and as it cuts through xy plane at $z=0$, this equipot value is also zero

$$V(R,\theta) = 0$$

$\vec{E}(\vec{r}) = E_0 \hat{z}$ as $\vec{r} \to \infty$ ← this is just boundary condition that sphere is in uniform applied field

integrate $\vec{E} = -\vec{\nabla} V$

$\Rightarrow V(\vec{r}) = -E_0 z + C$ but $C = 0$ since $V(x,y,0) = 0$

Finally: boundary conditions are:

1) $V(r,\theta) = -E_0 r \cos\theta$ as $r \to \infty$

2) $V(R,\theta) = 0$

Fit these b.c. to solution of form

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

2) $\Rightarrow$ $V(R,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} \right) P_\ell(\cos\theta) = 0$ for all θ

$$\Rightarrow A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = 0 \Rightarrow \boxed{B_\ell = -A_\ell R^{2\ell+1}}$$

1) $\Rightarrow$ $V(r \to \infty, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r \cos\theta$

(terms with $B_\ell \to 0$ as $r \to \infty$)

since RHS is linear in r, so must LHS be linear in r

$\Rightarrow A_\ell = 0$ except for $\ell = 1$

$$A_1 r P_1(\cos\theta) = -E_0 r \cos\theta$$

But $P_1(\cos\theta) = \cos\theta \Rightarrow A_1 = -E_0$

Solution is:

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

$$= -E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\cos\theta$$

potential due to applied field

potential due to induced surface charge on sphere (this is the dipole potential)

Induced charge density is

$$\frac{\sigma(\theta)}{\varepsilon_0} = \hat{m}\cdot\vec{E}(R,\theta) \Rightarrow \sigma(\theta) = -\varepsilon_0 \frac{\partial V(R,\theta)}{\partial r}$$

$$\sigma(\theta) = +\varepsilon_0 E_0\left(1 + \frac{2R^3}{r^3}\right)\Big|_{r=R}\cos\theta$$

$$= \varepsilon_0 E_0 (1+2)\cos\theta$$

$$\boxed{\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta}$$

$\sigma > 0$ for $\theta \in [0, \frac{\pi}{2})$
i.e. Northern hemisphere
$\sigma < 0$ for $\theta \in [\frac{\pi}{2}, \pi]$
i.e. Southern hemisphere.

$\vec{E}$ inside is $\vec{E}_{in} = \frac{-k}{3\varepsilon_0}\hat{z} = -\frac{3\varepsilon_0 E_0}{3\varepsilon_0}\hat{z} = -E_0\hat{z}$

Conducting sphere in a uniform applied field (second way)

$\vec{E}_0 = E_0 \hat{k} \Rightarrow$ potential $V_0(\vec{r}) = -E_0 z = -E_0 r\cos\theta$

We saw if there is (surface) charge density on sphere of $\sigma(\theta) = k\cos\theta$, then the resulting potential is

pot from $\sigma \rightarrow$ $$V_\sigma(r,\theta) = \begin{cases} \dfrac{k}{3\varepsilon_0} r\cos\theta & r<R \\ \dfrac{k}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta & r>R \end{cases}$$

If choose $k = 3\varepsilon_0 E_0$, then $V_\sigma + V_0 = 0$ for $r<R$
$\Rightarrow \vec{E}_{total} = 0$ inside sphere as it should be for conducting sphere. This then tells us that for the problem of the conducting sphere placed in a uniform applied $\vec{E}_0$, the induced surface charge is

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta.$$

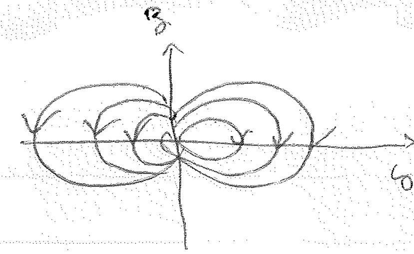
Now check the field outside conducting sphere

$$r>R \Rightarrow V_\sigma(r,\theta) = \frac{E_0 R^3}{r^2}\cos\theta \leftarrow \text{"dipole" potential}$$

find $\vec{E}_\sigma$, the electric field due to the (induced) surface charge

$$\vec{E}_\sigma = -\vec{\nabla} V_\sigma = -\frac{\partial V_\sigma}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_\sigma}{\partial \theta}\hat{\theta}$$

$$= \frac{2E_0 R^3 \cos\theta}{r^3}\hat{r} + \frac{E_0 R^3}{r^3}\sin\theta\,\hat{\theta}$$



$$\vec{E}_\sigma = \frac{E_0 R^3}{r^3}\left\{ 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right\}$$

this is the electric field of a dipole: decays as $1/r^3$
(compare to pt charge which decays as $1/r^2$)

for $\theta = 0, \pi, \ldots$ $\quad \vec{E}_\sigma = \pm 2\dfrac{E_0 R^3}{r^3}\,\hat{r}$

for $\theta = \frac{\pi}{2}$ $\quad \vec{E}_\sigma = \dfrac{E_0 R^3}{r^3}\,\hat{\theta}$

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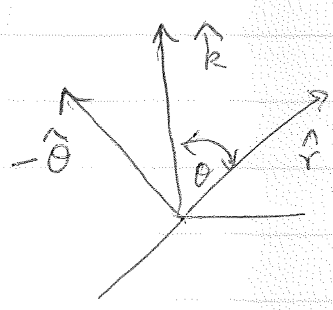
lets see that $\vec{E}_{tot}$ is <u>normal</u> to surface of conducting sphere

$$\vec{E}_{tot}(R,\theta) = E_0\hat{z} + \vec{E}_\sigma$$

on surface $\quad = E_0\hat{z} + \dfrac{E_0 R^3}{R^3}\left\{ 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right\}$



$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

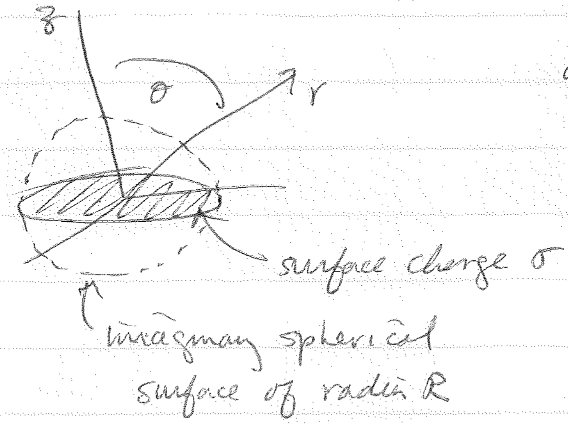
$$\vec{E}_{tot}(R,\theta) = \left[E_0\cos\hat{r} - E_0\sin\theta\,\hat{\theta} \right] \leftarrow \text{from } \vec{E}_0$$

$$+ \left[2E_0\cos\theta\,\hat{r} + E_0\sin\theta\,\hat{\theta} \right] \leftarrow \text{from } \vec{E}_\sigma$$

$$\boxed{\vec{E}_{tot} = 3E_0\cos\theta\,\hat{r}}$$

So we see that for $r = R$, ie on surface of sphere, $\vec{E}_{tot}$ is in radial direction, ie <u>normal</u> to the surface

Prob 3.22 Uniformly charged disk of radius R



along z axis, (see lecture 8)

$$V(r,\theta=0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right)$$

Find potential off z axis for $r>R$ and $r<R$!

Problem has ~~azi~~ symmetry in azimuthal angle ϕ, so can apply sep of var method.

Assume solution has form

$$V(r,\theta) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta) & r>R \\ \displaystyle\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r<R \end{cases}$$

For $r>R$, A_ℓ terms vanish as want $V\to 0$ as $r\to\infty$

For $r<R$, B_ℓ terms vanish as want V finite as $r\to 0$.

We have to find B_ℓ and A_ℓ subject to boundary condition on z-axis

$$V(r,0) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right)$$

$r>R$: $$V(r,0) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(1) \qquad \text{as } \cos 0 = 1$$

$$= \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} \qquad \text{as } P_\ell(1)=1$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2}-r\right)$$

to solve for B_ℓ, expand $\sqrt{\ }$ for small $\frac{R}{r}$

$$\sqrt{r^2+R^2} - r = r\sqrt{1+\frac{R^2}{r^2}} - r$$

use expansion $\sqrt{1+\epsilon} \cong 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \cdots$

(Taylor series about $\epsilon=0$)

$$\sqrt{r^2+R^2} - r \simeq r\left[1+\frac{1}{2}\left(\frac{R}{r}\right)^2 - \frac{1}{8}\left(\frac{R}{r}\right)^4 + \frac{1}{16}\left(\frac{R}{r}\right)^6 - \cdots\right] - r$$

$$\simeq \frac{1}{2}\frac{R^2}{r} - \frac{1}{8}\frac{R^4}{r^3} + \frac{1}{16}\frac{R^6}{r^5} - \cdots$$

$$\Rightarrow \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} = \frac{\sigma}{2\varepsilon_0}\left[\frac{R^2}{2r} - \frac{R^4}{8r^3} + \frac{R^6}{16r^5} - \cdots\right]$$

$$\Rightarrow \boxed{\begin{array}{ll} B_0 = \frac{\sigma}{2\varepsilon_0}\frac{R^2}{2}, & B_1 = 0 \\ B_2 = -\frac{\sigma}{2\varepsilon_0}\frac{R^4}{8}, & B_3 = 0 \\ B_4 = \frac{\sigma}{2\varepsilon_0}\frac{R^6}{16}, & B_5 = 0 \end{array}}$$

etc.

lowest two order term are: $V(r,\theta) = \underbrace{\frac{B_0}{r}P_0(\cos\theta)}_{\frac{\sigma R^2}{4\varepsilon_0 r} = \frac{\pi R^2\sigma}{4\pi\varepsilon_0 r}} + \frac{B_2}{r^3}P_2(\cos\theta)$

like from pt charge $q = \pi R^2 \sigma$

2nd order term : $\frac{B_2}{r^3} P_2(\cos\theta) = \underbrace{- \frac{\sigma R^4}{16 \varepsilon_0 r^3} \left[\frac{1}{2} \left(3\cos^2\theta - 1 \right) \right]}_{\text{quadrapole term.}}$

(dipole term would correspond to B_1, and this vanishes for uniformly charge disk)

$r < R$: $\theta = 0$; above disk

$$V(r,0) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(1) = \sum_{\ell=0}^{\infty} A_\ell r^\ell = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

expand $\sqrt{r^2+R^2} - r$ for small $\frac{r}{R}$

$$\sqrt{r^2+R^2} - r = R\sqrt{1 + \frac{r^2}{R^2}} - r$$

$$= R\left(1 + \frac{1}{2}\left(\frac{r}{R}\right)^2 - \frac{1}{8}\left(\frac{r}{R}\right)^4 + \frac{1}{16}\left(\frac{r}{R}\right)^6 \ldots\right) - r$$

$$= R - r + \frac{1}{2}\frac{r^2}{R} - \frac{1}{8}\frac{r^4}{R^3} + \frac{1}{16}\frac{r^6}{R^5}$$

$$A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + \ldots = \frac{\sigma}{2\varepsilon_0}\left[R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \frac{r^6}{16R^5}\right]$$

$$\Rightarrow \boxed{\begin{array}{l} A_0 = \frac{\sigma R}{2\varepsilon_0}, \quad A_1 = \frac{-\sigma}{2\varepsilon_0}, \quad A_2 = \frac{\sigma}{2\varepsilon_0}\frac{1}{2R} \\ A_3 = 0, \quad A_4 = \frac{-\sigma}{2\varepsilon_0}\frac{1}{8R^3}, \quad A_5 = 0, \quad A_6 = \frac{\sigma}{2\varepsilon_0}\frac{1}{16R^5} \end{array}}$$

etc. : the A_ℓ's alternat positive, zero, negative, zero, positive, ...

$r < R$: $\theta = \pi$ below disk Now use $P_\ell(-1) = \begin{cases} 1 & \ell \text{ even} \\ -1 & \ell \text{ odd} \end{cases}$

$$V(r,\pi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(-1) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{r^2+R^2} - r\right)$$

$$= A_0 - A_1 r + A_2 r^2 - A_3 r^3 + A_4 r^4 + \cdots$$

odd powers of ℓ change sign compared to $\theta = 0$ calculation

$$= \frac{\sigma}{2\varepsilon_0}\left[R - r + \frac{r^2}{2R} - \frac{r^4}{8R^3} + \cdots\right]$$

only even powers of r come here

$$\Rightarrow \boxed{A_0 = \frac{\sigma R}{2\varepsilon_0},\ A_1 = \frac{\sigma}{2\varepsilon_0},\ A_2 = \frac{\sigma}{2\varepsilon_0}\frac{1}{2R},\ A_3 = 0,\ A_4 = \frac{\sigma}{2\varepsilon_0 8R^3}}$$

all the A_ℓ's are the same as we found for above the disk, <u>except</u> for A_1 which changes sign.
This is <u>crucial</u> to get the correct jump in $\partial V/\partial z$ as one crosses the disk that we know must be there due to the charge density σ.

At the surface of the disk we must have

$$\left[-\frac{\partial V^{\text{above}}}{\partial z} + \frac{\partial V^{\text{below}}}{\partial z}\right]_{z=0} = \frac{\sigma}{\varepsilon_0}$$

Now compute the derivatives

$$= -\frac{\partial}{\partial z}\left[\sum_\ell A_\ell^{\text{above}} r^\ell P_\ell(\cos\theta) - \sum_\ell A_\ell^{\text{below}} r^\ell P_\ell(\cos\theta)\right]$$

$$= -\frac{\partial}{\partial z}\left[A_1^{\text{above}} r P_1(\cos\theta) - A_1^{\text{below}} r P_1(\cos\theta)\right]$$

since $A_\ell^{\text{above}} = A_\ell^{\text{below}}$ except for $\ell = 1$

$$= -\frac{\partial}{\partial z}\left[2 A_1^{above}\, r P_1(\cos\theta) \right] \qquad \text{since } A_1^{below} = -A_1^{above}$$

$$= -\frac{\partial}{\partial z}\left[2\left(\frac{-\sigma}{2\varepsilon_0}\right) r\cos\theta \right]$$

$$= \frac{\sigma}{\varepsilon_0}\frac{\partial}{\partial z}\left[r\cos\theta \right] \qquad \text{but } r\cos\theta = z$$

$$= \frac{\sigma}{\varepsilon_0}\frac{\partial}{\partial z}\left[z \right] = \frac{\sigma}{\varepsilon_0} \qquad \underline{\text{as it must be!}}$$

It is also interesting to compute the $\vec{E}$ field above the disk

$$\vec{E}_{above} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$V_{above}(r,\theta) = A_0 + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + A_3 r^3 P_3(\cos\theta)$$

$$= \frac{\sigma}{2\varepsilon_0}\left[R - r\cos\theta + \frac{r^2}{2R}\frac{1}{2}(3\cos^2\theta - 1) + 0 + \cdots \right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[R - r\cos\theta + \frac{r^2}{4R}(3\cos^2\theta - 1) + \cdots \right]$$

take derivatives to get

$$\vec{E}_{above} = \frac{\sigma}{2\varepsilon_0}\left[\left(\cos\theta - \frac{r}{2R}(3\cos^2\theta - 1)\right)\hat{r} + \left(-\sin\theta + \frac{3r}{2R}\cos\theta\sin\theta\right)\hat{\theta} \right]$$

Regroup terms

$$\vec{E}_{above} = \frac{\sigma}{2\varepsilon_0}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$+ \frac{\sigma}{2\varepsilon_0}\frac{r}{2R}\left[-(3\cos^2\theta - 1)\,\hat{r} + 3\cos\theta\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\hat{z} + \frac{\sigma}{4\varepsilon_0}\left(\frac{r}{R}\right)\left[-(3\cos^2\theta - 1)\hat{r} + 3\cos\theta\sin\theta\,\hat{\theta}\right]$$

where we used $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

The first term $\frac{\sigma}{2\varepsilon_0}\hat{z}$ is just the field one would get from an infinite flat plane!

The second term is the correction to that and goes like (r/R). The closer we want the $\vec{E}$ field to the edge of the disk, ie $r \to R$, the more terms in our Legendre series expansion we would have to consider. But for $r/R \ll 1$ we can get a good approx using only the above two terms.

The second term gives the correction resulting from the fact that the disk is NOT an infinite plane, but has a finite extent given by the radius R.