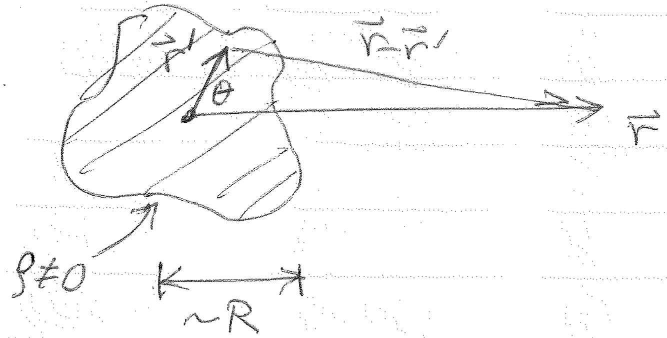
Multipole expansion

Potential at large distances $r \gg R$

position of observer (r)

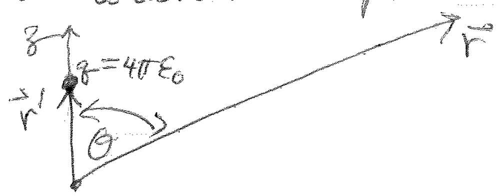
extent of charge distr (R)



$$V(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}')}{4\pi\varepsilon_0 \, |\vec{r}-\vec{r}'|}$$

We want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r \gg r'$.

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as potential at point $\vec{r}$ due to a charge $q = 4\pi\varepsilon_0$ located at position $\vec{r}'$ on the z axis



This problem has <u>polar</u> symmetry, ie. $V(r,\theta,\phi)$ is indep of polar angle ϕ as charge distr (pt charge at $\vec{r}'$) is symmetric with respect to rotations about z axis.

Therefore we can write $V(r,\theta)$ as expansion in Legendre polynomials, as in the charged disk problem

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$ — A_l terms = zero as need $V \to 0$ as $r \to \infty$

$$= \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} P_l(\cos\theta)$$

We know $V(r, \theta=0) = \dfrac{1}{r-r'}$ ← scalars here, since when $\theta=0$, $\vec{r}$ and $\vec{r}'$ are colinear

$$\Rightarrow V(r,0) = \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} P_l(1)$$

as $P_l(1) = 1$

$$= \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} = \frac{1}{r(1-r'/r)}$$

Now $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$ Taylor expansion for small ϵ

$$\Rightarrow \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots\right)$$

$\Rightarrow \boxed{B_l = (r')^l}$ is solution.

So for $r \gg r'$,

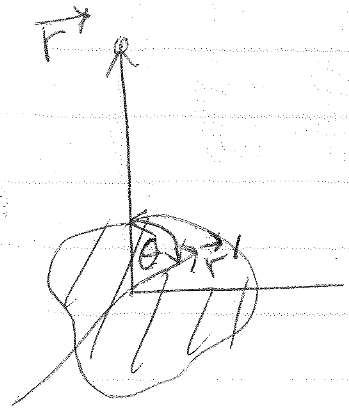
$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{r'}{r}\right)^l P_l(\cos\theta)}$$

So back to original problem:

$$V(\vec{r}) = \int d^3r' \left\{ \frac{\rho(\vec{r}\,')}{4\pi\varepsilon_0 r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta) \right\}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int_{\substack{\text{all}\\ \text{space}}} d^3r' \underbrace{\rho(\vec{r}\,')\,(r')^{\ell}\, P_\ell(\cos\theta)}_{\ell^{\text{th}}\text{ moment of charge distr.}}$$

where θ is the angle between the fixed observation pt $\vec{r}$, and the integration variable $\vec{r}\,'$.



This is the multipole expansion.
It expresses the "far" potential from a localized charge distribution as power series in (r'/r).
It is exact, provided one sums up all the infinite ℓ terms.
It is most often used as an approx by only summing up ℓ terms to some finite value.