Terms for $l=0,1,2$ are:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int d^3r'\rho(\vec{r}') \quad \leftarrow l=0,\text{ using } P_0(x)=1$$

"monopole" term $\sim \frac{1}{r}$

$$+\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int d^3r'\rho(\vec{r}')\,r'\cos\theta \quad \leftarrow l=1,\text{ using } P_1(x)=x$$

"dipole" term $\sim \frac{1}{r^2}$

$$+\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int d^3r'\rho(\vec{r}')(r')^2\tfrac{1}{2}(3\cos^2\theta-1) \quad \leftarrow l=2 \text{ using}$$

$$P_2(x)=\tfrac{1}{2}(3x^2-1)$$

"quadrupole" term $\sim \frac{1}{r^3}$

$l=3$ is "octopole" term

Recall θ in above integrals is angle between $\vec{r}$ of the observer and the integration variable $\vec{r}'$. We now want to reexpress these integrals in a way that lets us compute the above terms in terms of moments of the charge distribution that do not depend on the location of the observer at $\vec{r}$.

$l=0$ Monopole term: $\boxed{q \equiv \int d^3r'\rho(\vec{r}')}$ monopole moment q = total charge

$$\boxed{V_{mono}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}}$$

For a collection of point charges q_i at positions $\vec{r}_i$, $q=\sum_i q_i$

$\ell=1$ Dipole term:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \int d^3r' \rho(\vec{r}') r' \cos\theta$$

since $\vec{r}\cdot\vec{r}' = r r' \cos\theta$, we can write $\hat{r}\cdot\vec{r}' = r'\cos\theta$

then

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \hat{r}\cdot\left[\int d^3r' \rho(\vec{r}')\, \vec{r}'\right]$$

define dipole moment as the vector integral

$$\boxed{\vec{p} \equiv \int d^3r' \rho(\vec{r}')\, \vec{r}'}$$

then $$\boxed{V_{dip}(\vec{r}) = \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2}}$$

To compute $\vec{p}$ we in principle have to do 3 integrals

$$p_x = \int d^3r\, \rho(\vec{r})\, x$$
$$p_y = \int d^3r\, \rho(\vec{r})\, y$$
$$p_z = \int d^3r\, \rho(\vec{r})\, z$$

For a set of point charges q_i at positions $\vec{r}_i$

$$\vec{p} = \sum_i q_i \vec{r}_i$$

$l=2$ <u>Quadrupole term</u>:

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \int d^3r' \, (r')^2 \, \frac{1}{2}(3\cos^2\theta - 1)\, \rho(\vec{r}')$$

use $\cos\theta = \hat{r} \cdot \hat{r}'$

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \int d^3r' \rho(\vec{r}') \, \frac{1}{2}\left(3(\vec{r}'\cdot\hat{r})^2 - (r')^2\right)$$

$$= \frac{1}{4\pi\varepsilon_0 r^3} \, \hat{r} \cdot \left[\int d^3r' \, \rho(\vec{r}') \frac{1}{2}\left(3\vec{r}'\vec{r}' - (r')^2 \overleftrightarrow{I}\right)\right] \cdot \hat{r}$$

$\vec{r}'\vec{r}'$ is the <u>tensor product</u> of the two vectors
it is a 3×3 matrix whose components are

$$(\vec{r}\,\vec{r})_{ij} = \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix}$$

$\overleftrightarrow{I}$ is the identity tensor. It is a 3×3 matrix
whose components are just the Kronecker delta

$$(\overleftrightarrow{I})_{ij} \equiv \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To get a scalar from a tensor, one needs to
~~take~~ the dot product with a vector from each
side

$$\hat{r} \cdot (\vec{r}'\vec{r}') \cdot \hat{r} = (\hat{r}\cdot\vec{r}')(\vec{r}'\cdot\hat{r}) = (\vec{r}'\cdot\hat{r})^2$$

$$\hat{r} \cdot \overleftrightarrow{I} \cdot \hat{r} = \hat{r}\cdot\hat{r} = 1$$

We define the quadrupole tensor as

$$\boxed{\begin{aligned}\overleftrightarrow{\mathbb{Q}} &\equiv \int d^3r' \rho(\vec{r}')\left[3\vec{r}'\vec{r}' - (r')^2 \overleftrightarrow{I}\right] \\ V_{quad}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0 r^3}\,\frac{1}{2}\,\hat{r}\cdot\overleftrightarrow{\mathbb{Q}}\cdot\hat{r}\end{aligned}}$$

To compute the quadrupole tensor $\overleftrightarrow{\mathbb{Q}}$ we in principle have to do 9 integrals to compute the 9 components of the 3×3 matrix

$$\mathbb{Q}_{ij} = \int d^3r' \rho(\vec{r}')\left[3 r_i' r_j' - (r')^2 \delta_{ij}\right]$$

But note that $\overleftrightarrow{\mathbb{Q}}$ is a symmetric matrix, ie $\mathbb{Q}_{ij} = \mathbb{Q}_{ji}$ so there are really only 6 integrals to do

$$\mathbb{Q}_{ij} = \begin{pmatrix} \mathbb{Q}_{xx} & \mathbb{Q}_{xy} & \mathbb{Q}_{xz} \\ \mathbb{Q}_{yx} & \mathbb{Q}_{yy} & \mathbb{Q}_{yz} \\ \mathbb{Q}_{zx} & \mathbb{Q}_{zy} & \mathbb{Q}_{zz} \end{pmatrix}$$

with

$$\mathbb{Q}_{xx} = \int d^3r \left(3x^2 - (x^2+y^2+z^2)\right) \rho(\vec{r})$$

$$= \int d^3r \left(2x^2 - y^2 - z^2\right) \rho(\vec{r})$$

$$\mathbb{Q}_{yy} = \int d^3r \left(2y^2 - x^2 - z^2\right) \rho(\vec{r})$$

$$\mathbb{Q}_{zz} = \int d^3r \left(2z^2 - x^2 - y^2\right) \rho(\vec{r})$$

$$Q_{xy} = Q_{yx} = \int d^3r \; 3xy \; \rho(\vec{r})$$

$$Q_{xz} = Q_{zx} = \int d^3r \; 3xz \; \rho(\vec{r})$$

$$Q_{yz} = Q_{zy} = \int d^3r \; 3yz \; \rho(\vec{r})$$

For a set of point charges q_i at positions $\vec{r}_i$ we have

$$Q_{xx} = \sum_i q_i \left(2x_i^2 - y_i^2 - z_i^2\right)$$

$$Q_{xy} = 3\sum_i q_i x_i y_i$$

etc.

In the multipole expansion we approximate the potential $V(\vec{r})$ from a charge distribution $\rho(\vec{r})$ in terms of the moments of $\rho(\vec{r})$. The three lowest order terms give:

$$V(\vec{r}) = \underbrace{\frac{q}{4\pi\varepsilon_0 r}}_{\text{monopole term}} + \underbrace{\frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2}}_{\text{dipole term}} + \underbrace{\frac{\frac{1}{2}\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{4\pi\varepsilon_0 r^3}}_{\text{quadrupole term}} + \cdots$$

$$q \equiv \int d^3r \; \rho(\vec{r})$$

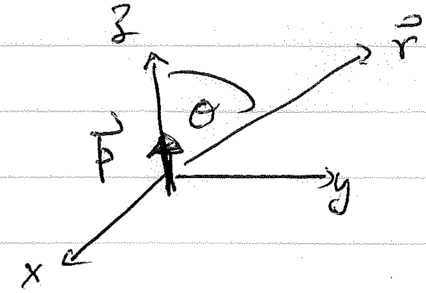
$$\vec{p} \equiv \int d^3r \; \rho(\vec{r})\,\vec{r}$$

$$\overleftrightarrow{Q} \equiv \int d^3r \; \rho(\vec{r}) \left[3\vec{r}\,\vec{r} - r^2 \overleftrightarrow{I}\right]$$

we only need to compute the integrals q, $\vec{p}$, $\overleftrightarrow{Q}$, once to get the approx for $V(\vec{r})$ at any position $\vec{r}$.

Dipole term

$$V_{dip}(\vec{r}) = \frac{\hat{r}\cdot\vec{p}}{4\pi\varepsilon_0 r^2}$$



It is often convenient to express V_{dip} in spherical coordinates where we choose the $\hat{z}$ axis to be parallel to the dipole moment $\vec{p}$.

Then $\hat{r}\cdot\vec{p} = p\cos\theta$

$$V_{dip}(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

does not depend on φ since rotational symmetry about z axis

We can now compute the electric field $\vec{E}$ from this dipole term

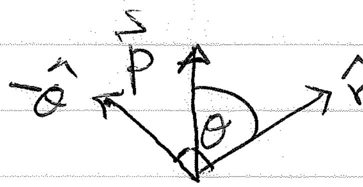
$$\vec{E}_{dip} = -\vec{\nabla}V_{dip} = -\frac{\partial V_{dip}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_{dip}}{\partial\theta}\hat{\theta}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

This is exactly the same $\vec{E}$ as we found for the conducting sphere with surface charge $\sigma(\theta) = k\cos\theta$

It is useful to have an expression for $\vec{E}_{dip}$ that does not depend on choosing a particular coordinate system (ie such that $\vec{p}$ is parallel to $\hat{y}$).

We can derive such an expression as follows:



$$\Rightarrow \quad p\cos\theta\,\hat{r} = (\vec{p}\cdot\hat{r})\,\hat{r}$$

$$p\sin\theta\,\hat{\theta} = (-\vec{p}\cdot\hat{\theta})\,\hat{\theta}$$

also $\quad \vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$

so $\quad (-\vec{p}\cdot\hat{\theta})\hat{\theta} = (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$

so $\quad 2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}$

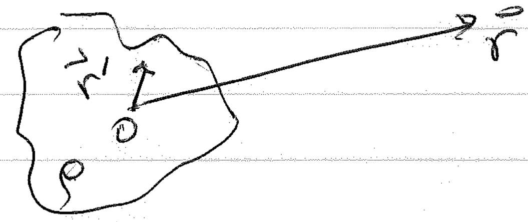
$$= 2(\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$$

$$= 3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}$$

$$\boxed{\vec{E}_{dip} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

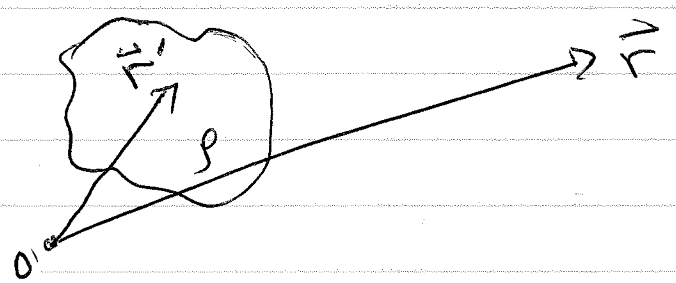
expresses $\vec{E}_{dip}$ without reference to any particular coordinate system.

Moments and the origin of the coordinate system



When we developed the multipole expansion, we said we put the origin of our coordinates somewhere in the middle of the localized charge distribution.

But where exactly should we put the origin. Clearly it would make a big difference if we made a <u>bad</u> choice for the origin

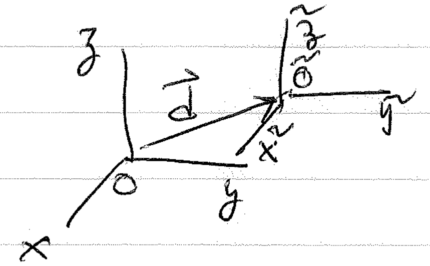


The location of the origin of our coordinates <u>will</u> effect the values of the charge distribution moments. We can sometimes improve our multipole expansion by a clever choice of where to put the origin when we are computing the moments of the charge distribution.

effect of origin on dipole moment

$$\vec{p} = \int_{\substack{\text{all}\\\text{space}}} d^3r\, \rho(\vec{r})\, \vec{r}$$

suppose we shift to new coordinates by displacing the origin a distance $\vec{d}$.



New coordinates $\tilde{\vec{r}} = \vec{r} - \vec{d}$

Then in this new coordinate system

$$\tilde{\vec{p}} = \int d^3\tilde{r}\, \rho(\tilde{\vec{r}})\, \tilde{\vec{r}}$$

$$= \int d^3r\, \rho(\vec{r})\,(\vec{r} - \vec{d}) = \int d^3r\, \rho(\vec{r})\,\vec{r} - \vec{d}\int d^3r\, \rho(\vec{r})$$

$$= \vec{p} - \vec{d}\, q$$

So in general the dipole moment $\tilde{\vec{p}}$ computed in the $\tilde{\vec{r}}$ coordinate system will be different from the dipole moment $\vec{p}$ computed in the $\vec{r}$ coordinate system.

They will only be equal, ie $\tilde{\vec{p}} = \vec{p}$, when $q = 0$ ie if the charge distribution $\rho(\vec{r})$ has no net charge.

When $q = 0$, the dipole moment $\vec{p}$ has the same value in any coordinate system!

But suppose $q \neq 0$, then we can always make a choice of the location of the origin, so that in that coordinate system the dipole moment will vanish!

If $q \neq 0$, compute $\vec{p}$ about some origin. In general it will be a finite value. Now define the displacement vector

$$\vec{d} \equiv \frac{\vec{p}}{q} \quad \text{where } q \text{ is the total charge}$$

then if we transform to a new coordinate system

$\tilde{\vec{r}} = \vec{r} - \vec{d}$, then $\tilde{\vec{p}}$ in this new coordinate system will be

$$\tilde{\vec{p}} = \vec{p} - \vec{d}q = \vec{p} - \left(\frac{\vec{p}}{q}\right)q = 0 \;!$$

So in the $\tilde{\vec{r}}$ coord system, the dipole moment always vanishes! So this is the best choice of origin to use for the multipole expansion since in this case the dipole term vanishes and one gets

$$V(\vec{r}) \simeq \frac{q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^3}\frac{1}{2}\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}$$

dipole term $\sim \frac{1}{r^2}$ vanishes

leading correction to monopole term is now the quadrupole $\sim \frac{1}{r^3}$ rather than the dipole $\sim 1/r^2$

effect of shift of origin on quadrupole tensor

$$Q_{ij} = \int d^3r \, (3 r_i r_j - r^2 \delta_{ij}) \, \rho(\vec{r})$$

shift origin to new coord system $\tilde{\vec{r}} = \vec{r} - \vec{d}$

$$\tilde{Q}_{ij} = \int d^3\tilde{r} \, (3 \tilde{r}_i \tilde{r}_j - \tilde{r}^2 \delta_{ij}) \, \rho(\vec{r})$$

$$= \int d^3r \, \big(3 (r_i - d_i)(r_j - d_j) - (\vec{r} - \vec{d}) \cdot (\vec{r} - \vec{d}) \, \delta_{ij} \big) \rho(\vec{r})$$

$$= \int d^3r \Big(3 r_i r_j - 3 r_i d_j - 3 r_j d_i + 3 d_i d_j$$

$$- (r^2 + d^2 - 2 \vec{r} \cdot \vec{d}) \, \delta_{ij} \Big) \rho(\vec{r})$$

$$= \int d^3r \, (3 r_i r_j - r^2 \delta_{ij}) \, \rho(\vec{r})$$

$$- \int d^3r \left[(3 r_i d_j + 3 r_j d_i) - 2 \vec{r} \cdot \vec{d} \, \delta_{ij} \right] \rho(\vec{r})$$

$$+ \int d^3r \left[3 d_i d_j - d^2 \delta_{ij} \right] \rho(\vec{r})$$

$$\tilde{Q}_{ij} = Q_{ij} - 3 d_j p_i - 3 d_i p_j + 2 \vec{d} \cdot \vec{p} \, \delta_{ij}$$

$$+ (3 d_i d_j - d^2 \delta_{ij}) \, q$$

So $\tilde{Q}_{ij} = Q_{ij}$ <u>only if</u> <u>both</u> $q = 0$ and $\vec{p} = 0$