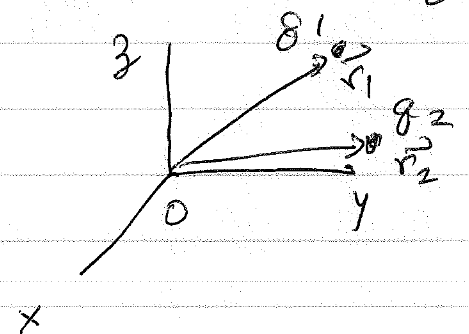
Example q_1 at $\vec{r}_1$, q_2 at $\vec{r}_2$

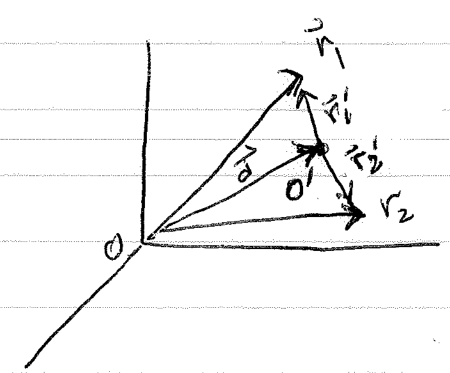


monopole $q = q_1 + q_2$

dipole $\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$

quadrupole $$Q_{ij} = (3 r_{1i} r_{1j} - r_1^2 \delta_{ij}) q_1 + (3 r_{2i} r_{2j} - r_2^2 \delta_{ij}) q_2$$

case ①: Suppose $q = q_1 + q_2 \neq 0$



$$\vec{p} = q_1 \vec{r}_1 + q_2 \vec{r}_2$$

define $\vec{d} = \frac{\vec{p}}{q} = \frac{q_1 \vec{r}_1 + q_2 \vec{r}_2}{q_1 + q_2}$

define $\vec{r}' = \vec{r} - \vec{d}$

then $$\vec{p}' = q_1 \vec{r}_1' + q_2 \vec{r}_2' = q_1(\vec{r}_1 - \vec{d}) + q_2(\vec{r}_2 - \vec{d})$$
$$= q_1 \vec{r}_1 + q_2 \vec{r}_2 - (q_1 + q_2)\vec{d}$$
$$= \vec{p} - q\vec{d} = 0$$

$$\vec{r}_1' = \vec{r}_1 - \frac{\vec{p}}{q} = \vec{r}_1 - \frac{(q_1 \vec{r}_1 + q_2 \vec{r}_2)}{q_1 + q_2} = \frac{q_2(\vec{r}_1 - \vec{r}_2)}{q_1 + q_2}$$

$$\vec{r}_2' = \vec{r}_2 - \frac{\vec{p}}{q} = \vec{r}_2 - \frac{(q_1 \vec{r}_1 + q_2 \vec{r}_2)}{q_1 + q_2} = \frac{q_1(\vec{r}_2 - \vec{r}_1)}{q_1 + q_2}$$

$\vec{r}' = 0$ is "center of charge"

In the $\vec{r}'$ coordinate system, where $\vec{p}' = 0$, lets compute $Q'_{ij} = (3r'_{1i}r'_{1j} - (r'_1)^2\delta_{ij})\,q_1$

$$+ (3r'_{2i}r'_{2j} - (r'_2)^2\delta_{ij})\,q_2$$

Define $\delta\vec{r} = \vec{r}_1 - \vec{r}_2$ so $\vec{r}_1{}' = \frac{q_2}{q_1+q_2}\,\delta\vec{r}$

$$\vec{r}_2{}' = \frac{q_1}{q_1+q_2}(-\delta\vec{r})$$

$$Q'_{ij} = \left(3\,\delta r_i\,\delta r_j - \delta r^2\delta_{ij}\right)\frac{q_1 q_2^2}{(q_1+q_2)^2}$$

$$+ \left(3\delta r_i\,\delta r_j - \delta r^2\delta_{ij}\right)\frac{q_2 q_1^2}{(q_1+q_2)^2}$$

$$= \left(3\delta r_i\,\delta r_j - \delta r^2\delta_{ij}\right)\left[\frac{q_1 q_2^2 + q_2 q_1^2}{(q_1+q_2)^2}\right]$$

$$= \left(3\delta r_i\,\delta r_j - \delta r^2\,\delta_{ij}\right)\frac{q_1 q_2}{q_1+q_2}$$

Choose coordinate system so the $\delta\vec{r}$ is parallel to $\hat{z}$

$\delta\vec{r} = s\hat{z}$ s is distance between charges

$$\Rightarrow \delta r_x = \delta r_y = 0 \qquad \delta r_z = s$$

$$\delta r^2 = s^2$$

Then

$$Q'_{ij} = \frac{q_1 q_2}{q_1+q_2}\begin{pmatrix} 3\delta r_x^2 - \delta r^2 & 3\delta r_x \delta r_y & 3\delta r_x \delta r_z \\ 3\delta r_x \delta r_y & 3\delta r_y^2 - \delta r^2 & 3\delta r_y \delta r_z \\ 3\delta r_x \delta r_z & 3\delta r_y \delta r_z & 3\delta r_z^2 - \delta r^2 \end{pmatrix}$$

$$= \frac{q_1 q_2}{q_1+q_2}\begin{pmatrix} -s^2 & 0 & 0 \\ 0 & -s^2 & 0 \\ 0 & 0 & 2s^2 \end{pmatrix}$$

$$Q'_{ij} = \frac{q_1 q_2 s^2}{q_1+q_2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

potential is:

$$V = \frac{q}{4\pi\varepsilon_0 r} + \frac{\frac{1}{2}\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r}}{4\pi\varepsilon_0 r^3}$$

use $\hat{r} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$

$$\overleftrightarrow{Q}\cdot\hat{r} = \frac{q_1 q_2 s^2}{q_1+q_2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

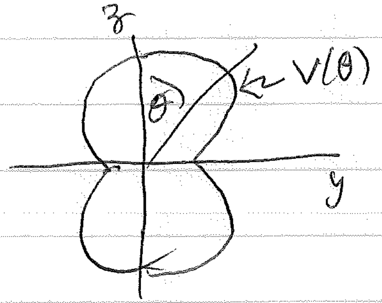
$$= \frac{q_1 q_2 s^2}{q_1+q_2}\begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ 2\cos\theta \end{pmatrix}$$

$$\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r} = \frac{q_1 q_2 s^2}{q_1+q_2}\left(-\sin^2\theta\cos^2\varphi - \sin^2\theta\sin^2\varphi + 2\cos^2\theta\right)$$

$$= \frac{q_1 q_2 s^2}{q_1+q_2}\left(2\cos^2\theta - \sin\theta\right)$$

$$\underbrace{\qquad\qquad}^{\frac{1}{2}(3\cos 2\theta + 1)}$$

$$V = \frac{q}{4\pi\varepsilon_0 r} + \frac{\delta^2}{8\pi\varepsilon_0 r^3}\left(\frac{q_1 q_2}{q_1+q_2}\right)(2\cos^2\theta - \sin^2\theta)$$



as measured in coordinate system where $\vec{p}\,' = 0$ ie origin is at the "center of charge"

case ② Suppose $q = q_1 + q_2 = 0$ ie $q_2 = -q_1$

then $\vec{p} = q_1\vec{r}_1 + q_2\vec{r}_2 = q_1(\vec{r}_1 - \vec{r}_2)$

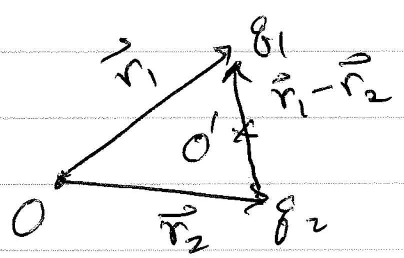
independent of where we put the origin of our coordinate system,

Can we choose the origin in some clever way to minimize the contribution to V from the quadrupole term?

using $q_2 = -q_1$

$$Q_{ij} = \left[(3r_{1i}r_{1j} - r_1^2\delta_{ij}) - (3r_{2i}r_{2j} - r_2^2\delta_{ij})\right]q_1$$

If we choose the origin to lie midway between the two charges then in this new coord system



$$\vec{r}_1\,' = \vec{r}_1 - \vec{d} = \vec{r}_1 - \frac{(\vec{r}_1+\vec{r}_2)}{2} = \frac{\vec{r}_1-\vec{r}_2}{2}$$

$$\vec{r}_2\,' = \vec{r}_2 - \vec{d} = \vec{r}_2 - \frac{(\vec{r}_1+\vec{r}_2)}{2} = \frac{\vec{r}_2-\vec{r}_1}{2} = -\vec{r}_1\,'$$

displacement from O to O' is $\vec{d} = \dfrac{\vec{r}_1+\vec{r}_2}{2}$

Then $Q'_{ij} = \Big[(3 r'_{1i} r'_{1j} - (r_1')^2 \delta_{ij}$

using $\vec{r}_2' = -\vec{r}_1'$ $\quad - (3(-r'_{1i})(-r'_{1j}) - (-r_1')^2 \delta_{ij} \Big] q_1$

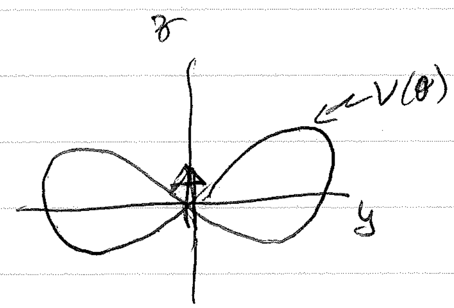
$$Q'_{ij} = \left[(3 r'_{1i} r'_{1j} - (r_1')^2 \delta_{ij}) - (3 r'_{1i} r'_{1j} - (r_1')^2 \delta_{ij}) \right] q_1$$

$$= 0$$

So quadrupole moment vanishes!

$$V(\vec{r}) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2} + \underset{\text{term}}{\text{octopole}} \; O\left(\frac{1}{r^4}\right)$$

Note: it worked in this particular example that we could choose the origin so that $\overleftrightarrow{Q}' = 0$. For a general charge distribution where $q = 0$ it is usually <u>not</u> possible to do this and $\overleftrightarrow{Q}$ will stay finite. But still it may be possible to choose the origin so that $\overleftrightarrow{Q}$ is as small as it can be.

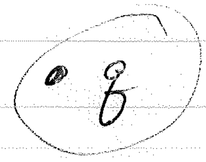


dipole potential polar plot

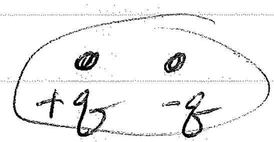
$$V_{dip} = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} = \frac{p \cos\theta}{4\pi\epsilon_0 r^2}$$

Examples of charge distribution with monopole, dipole, and quadrupole moments

monopole

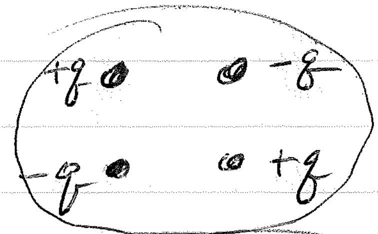


dipole



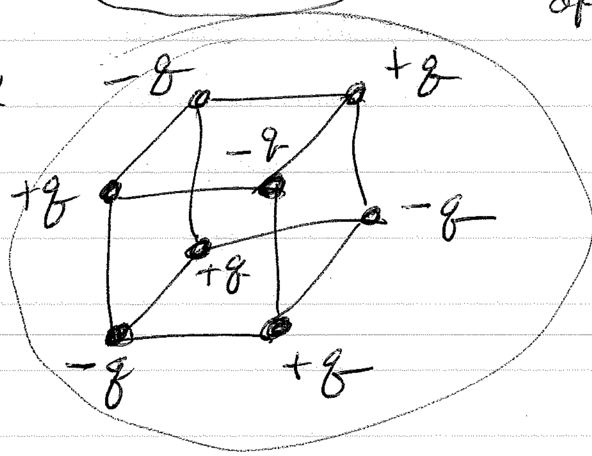
monopole = 0

quadrupole



monopole = 0
dipole = 0
can view as a pair of oppositely oriented dipoles

octopole



monopole = 0
dipole = 0
quadrupole = 0
can view as a pair of oppositely oriented quadrupoles

Dielectric materials (insulators)

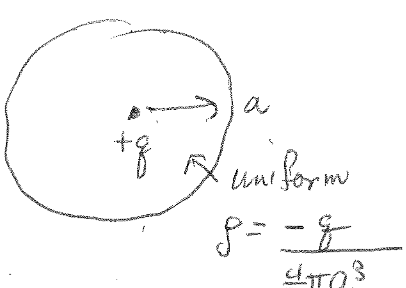
<u>conductors</u>: electrons free – when apply $\vec{E}_{appl}$ electrons move to set up equal and opposite $\vec{E}_{ind}$ so that total $\vec{E}_{tot} = \vec{E}_{appl} + \vec{E}_{ind} = 0$.

<u>insulator = dielectric</u>: electrons bound to atomic nuclei – can't move freely, but they can move a little bit and give a net dipole moment to material. Sum of all atomic dipole moments which get induced when apply $\vec{E}_{appl}$ give rise to electric field $\vec{E}_{ind}$ that only partially cancells $\vec{E}_{appl}$. $\vec{E}_{tot} \neq 0$ for dielectric

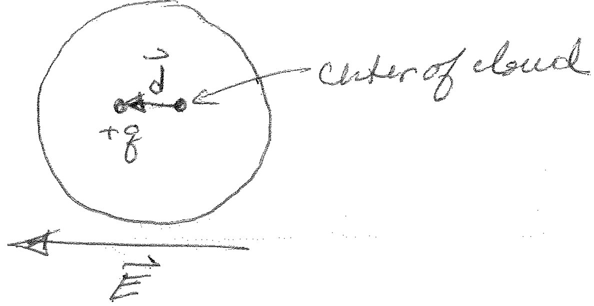
<u>polarizability</u>: when a neutral atom is placed in external electric field, it can develope a dipole moment $\vec{p}$ proportional to $\vec{E}$.

$$\vec{p} = \underbrace{\alpha}_{\text{atomic polarizability}} \vec{E}$$

<u>Simple model of atom</u>: proton surrounded by electron cloud

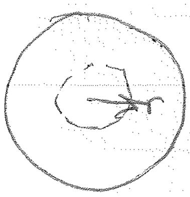


apply $\vec{E}$, and center of cloud + proton seperate a distance $\vec{d}$.



What is force on proton due to electron?

inside uniformly charged sphere $\vec{E}(\vec{r}) = \frac{\rho r}{3\varepsilon_0}\hat{r}$



$$4\pi r^2 E(r) = \frac{\rho \frac{4}{3}\pi r^3}{\varepsilon_0} \qquad = \frac{-q}{\frac{4}{3}\pi a^3}\frac{r}{3\varepsilon_0}\hat{r} = \frac{-q\, r}{4\pi\varepsilon_0 a^3}\hat{r}$$

$$E(r) = \frac{\rho r}{3\varepsilon_0}$$

force from electron is $\vec{F}_e = +q\vec{E}(\vec{d}) = \frac{-q^2\vec{d}}{4\pi\varepsilon_0 a^3}$

force from applied electric field is

$$\vec{F}_{appl} = +q\vec{E}_{appl}$$

proton comes to rest at position $\vec{d}$ such that $\vec{F}_e + \vec{F}_{appl} = 0$
net force is zero

$$\Rightarrow \quad q\vec{E} - \frac{q^2\vec{d}}{4\pi\varepsilon_0 a^3} = 0 \quad \Rightarrow \quad \vec{d} = \frac{4\pi\varepsilon_0 a^3 \vec{E}}{q}$$

net dipole moment is $\vec{p} = q\vec{d} = 4\pi\varepsilon_0 a^3\vec{E}$

$$\Rightarrow \quad \boxed{\alpha = 4\pi\varepsilon_0 a^3}$$