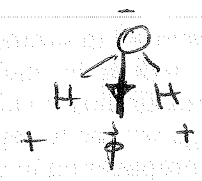
For molecule with <u>intrinsic dipole</u> moment even when $\vec{E}=0$

ex:



<u>Force on molecule</u>:

$$\vec{F} = \int d^3r\, \rho(\vec{r})\, \vec{E}(\vec{r})$$

i^{th} component $\quad F_i = \int d^3r\, \rho(\vec{r})\, E_i(\vec{r}) \qquad$ $\vec{r}_0$ in center of molecule

For slowly varying $\vec{E}(\vec{r})$, $\quad E_i(\vec{r}) \simeq E_i(\vec{r}_0) + \sum_j \frac{\partial E_i(\vec{r}_0)}{\partial r_j}(r_j - r_{0_j})$

$$F_i = \int d^3r\, \rho(\vec{r})\, \boxed{E_i(\vec{r}_0)} + \sum_j \frac{\partial E_i(\vec{r}_0)}{\partial r_j} \int d^3r\, \rho(\vec{r})(r_j - r_{0_j})$$

For neutral molecule $\int d^3r\, \rho = 0$

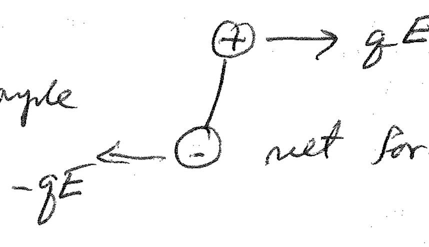
$$F_i = 0 + \sum_j \frac{\partial E_i}{\partial r_j} \int d^3r\, \rho(\vec{r})\, r_j = \sum_j \frac{\partial E_i}{\partial r_j}\, p_j$$

$$= (\vec{p}\cdot\vec{\nabla})E_i$$

$$\boxed{\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}} \qquad \vec{p} \text{ is dipole moment}$$

If $\vec{E}$ is uniform, then $\vec{F} = 0$

example

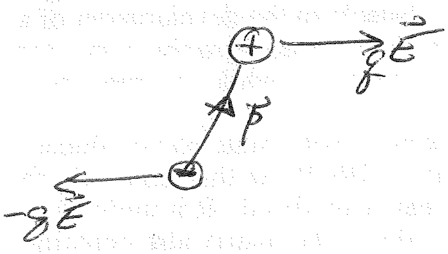


net force $= 0$

Torque on molecule in uniform $\vec{E}$ field

$$\vec{N} = \int d^3r \; \vec{r} \times \vec{f}(\vec{r}) = \int d^3r \; \vec{r} \times \rho(\vec{r}) \vec{E}$$

$$= \left[\int d^3r \; \rho(\vec{r}) \, \vec{r} \right] \times \vec{E} = \boxed{\vec{p} \times \vec{E} = \vec{N}}$$



torque tries to allign dipole parallel to $\vec{E}$

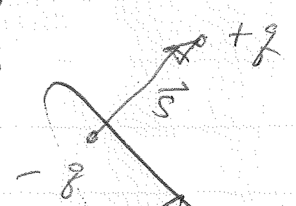
When $\vec{E} = 0$, molecules randomly oriented, $\Rightarrow$ net dipole moment averaged on microscopic scale $= 0$.

When apply $\vec{E}$, all dipoles reorient slightly more in direction of $\vec{E}$ $\Rightarrow$ gives rise to net dipole moment

$$\vec{p} = \alpha \vec{E}$$

↑ molecular polarizability

Aside: for non uniform $\vec{E}$, force on dipole is



$$\vec{F} = q\vec{E}(\vec{r}_+) - q\vec{E}(\vec{r}_-)$$

$$\vec{r}_+ = \vec{r}_- + \vec{s}$$

$$\vec{E}(\vec{r}_+) \simeq \vec{E}(\vec{r}_-) + \sum_{i=1}^{3} \left.\frac{\partial \vec{E}}{\partial r_i}\right|_{\vec{r}=\vec{r}_-} s_i$$

$$\vec{F} = q\vec{E}(\vec{r}_-) + q\sum_{i=1}^{3} \frac{\partial \vec{E}}{\partial r_i} s_i - q\vec{E}(\vec{r}_-)$$

$$= \sum_{i=1}^{3} \underbrace{q s_i}_{p_i} \frac{\partial \vec{E}}{\partial r_i} = \boxed{(\vec{p}\cdot\vec{\nabla})\vec{E} = \vec{F}}$$

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Henceforth, forget the microscopic mechanism that causes medium to develope local dipole moments. Just assume it does

$$\vec{p} = \alpha \vec{E}$$

Define polarization $\vec{\mathbb{P}}$ = dipole moment per unit volume

$$\vec{\mathbb{P}}(\vec{r}) = \sum_i \vec{p}_i\, \delta(\vec{r}-\vec{r}_i)$$

dipole moment of atom i, located at position $\vec{r}_i$

$$\int d^3r\, \vec{\mathbb{P}}(\vec{r}) = \int d^3r \sum_i \vec{p}_i\, \delta(\vec{r}-\vec{r}_i) = \sum_i \vec{p}_i = \vec{p}_{total}$$

total dipole moment on material.

Consider, if average on length scales small compared to size of system (macroscopic) but big compared to atom (microscopic)

then $\vec{P}(\vec{r})$ is smooth continuous function - just like view charge density $\rho(\vec{r})$ as smooth + continuous,

What is potential $V(\vec{r})$ created by having a non zero polarization density $\vec{P}(\vec{r})$?

For single dipole at <u>origin</u>, $V(\vec{r}) = \dfrac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$

For dipole at $\vec{r}\,'$ $\quad V(\vec{r}) = \dfrac{\vec{p}\cdot(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3}$

$\Rightarrow$ from polarization density

$$V(\vec{r}) = \int_{\text{volume}} d^3r' \, \frac{\vec{P}(\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3}$$

use $-\vec{\nabla}\left(\dfrac{1}{|\vec{r}-\vec{r}\,'|}\right) = \dfrac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$

derivatives wrt $\vec{r}$

$\qquad \parallel$

derivatives wrt $\vec{r}\,'$ $\quad \vec{\nabla}'\left(\dfrac{1}{|\vec{r}-\vec{r}\,'|}\right)$

$$V(\vec{r}) = \int \frac{d^3r'}{4\pi\varepsilon_0} \vec{P}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

integrate by parts $\quad \vec{\nabla}\cdot(\vec{F}g) = (\vec{\nabla}\cdot\vec{F})g + \vec{F}\cdot\vec{\nabla}g$

$$= \int \frac{d^3r'}{4\pi\varepsilon_0}\left\{ \vec{\nabla}'\cdot\left(\frac{\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right) - \left(\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')\right)\frac{1}{|\vec{r}-\vec{r}\,'|}\right\}$$

use Gauss law

$$V(\vec{r}) = \int_{\substack{\text{surface} \\ \vec{r}' \text{ on surface}}} \frac{d\vec{a}' \cdot \vec{P}(\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|} + \int_{\substack{\text{vol} \\ \vec{r}' \in \text{vol}}} \frac{d^3r'}{4\pi\varepsilon_0} \frac{(-)\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

looks like potential from charge density $\boxed{\rho_b(\vec{r}') \equiv -\vec{\nabla} \cdot \vec{P}(\vec{r}')}$

looks like potential from surface charge density $\boxed{\sigma_b(\vec{r}') = \hat{n} \cdot \vec{P}(\vec{r}')}$ $\quad d\vec{a} = da\,\hat{n}$

$\vec{r}'$ pt on surface

$$\boxed{V(\vec{r}) = \int da' \frac{\sigma_b(\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|} + \int d^3r' \frac{\rho_b(\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}}$$

$\Rightarrow$ a polarized medium induces a "bound" charge density $\rho_b = -\vec{\nabla} \cdot \vec{P}$
and a "bound" surface charge density $\sigma_b = \hat{n} \cdot \vec{P}$
where $\hat{n}$ is outward pointing normal at surface of material

ex: uniformly polarized sphere: choose $\vec{P}$ along $\hat{z}$

$\vec{\nabla} \cdot \vec{P} = 0$ inside as P uniform $\Rightarrow \rho_b = 0$

$\uparrow P$

$$\sigma_b = \hat{n} \cdot \vec{P} = \hat{r} \cdot \hat{k}\, P = P \cos\theta$$

we already solved the problem of fields from a sphere with surface charge $\sigma(\theta) = P\cos\theta$

(back when we did this, we called the constant $\underset{P}{\overset{k}{\parallel}}$)

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\varepsilon_0}\, r\cos\theta & r < R \\[2ex] \dfrac{P}{3\varepsilon_0}\dfrac{R^3}{r^2}\cos\theta & r \geq R \end{cases}$$

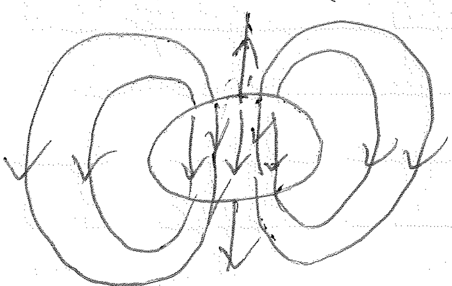
$= \dfrac{P}{3\varepsilon_0} z \Rightarrow$ uniform $\vec{E} = \dfrac{-\vec{P}}{3\varepsilon_0}$

$\leftarrow$ dipole potential

$$= \frac{\frac{4}{3}\pi R^3 P\cos\theta}{\frac{4}{3}\pi\, 3\varepsilon_0 r^2}$$

$$= \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

where $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$ total dipole moment on sphere



Note: Net ~~charges~~ "bound" charges induced must sum to zero as dielectric is neutral

$$\Rightarrow \int da\, \sigma_b + \int d^3r\, \rho_b = 0$$

follows from $\displaystyle \int d\vec{a}\cdot\vec{P} + \underbrace{\int d^3r\,(-\vec{\nabla}\cdot\vec{P})}_{= -\int d\vec{a}\cdot\vec{P} \text{ by Gauss theorem}} = 0$

Note that for this example, $V(\vec{r})$ as obtained above in dipole approx gives exact answer both inside and outside the sphere - very important result for later

Note: When we write

$$V(\vec{r}) = \int_{\text{surface}} da' \frac{\sigma_b(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|} + \int_{\text{vol}} d^3r \frac{\rho_b(\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|}$$

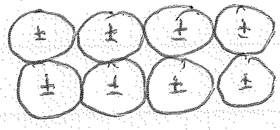
with $\sigma(\vec{r}) = \hat{n} \cdot \mathbb{P}(\vec{r})$ on surface
$\rho_b(\vec{r}) = -\vec{\nabla} \cdot \mathbb{P}(\vec{r})$ inside dielectric

we used the dipole approximation to compute the potential (and hence the electric field) from the dipoles in the polarized material.

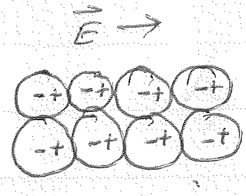
But the dipole approximation was an approximation that was supposed to be good when ~~one~~ the observer is far away from the dipole. Here we have used it to also give the $\vec{E}$ field inside the dielectric i.e., close to the dipoles that give rise to the fields. Why are we able to do this? Why don't we need to consider the fields produced by the quadrupole moments, etc, of the polarized atoms + molecules?

The answer has to do with the observation that the dipole approximation turns out to give the exact correct $\vec{E}$ field inside the uniformly polarized dielectric sphere. To see the complete argument read Griffiths section 4.2.3.

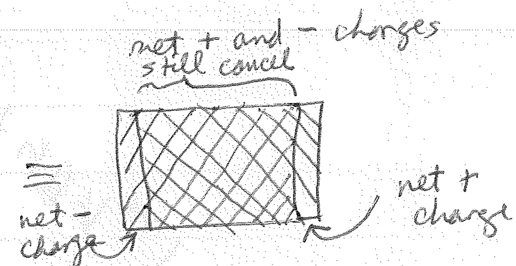
Bound charges are real charges



dielectric with $\vec{E}=0$



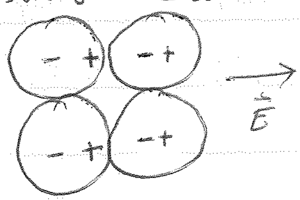
polarized dielectric in $\vec{E}$ uniform $\Rightarrow \vec{P}$ uniform



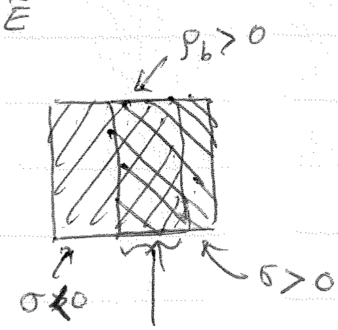
$\equiv$ region of + charge shifts to right, relative to region of - charge builds up surface charge $\sigma_b > 0$ on right, $\sigma_b < 0$ on left $\sigma_b = \hat{n}\cdot\vec{P}$

For non-uniform field $\vec{E}$

E strong E weak



separation between (- +) is larger in region of larger $\vec{E}$



(+) charges compressed into smaller region than (-) charges
$\Rightarrow$ in region of overlap, there is net + charge density

Net bound charge must sum to zero

$$\int_{vol} d^3r\, \rho_b + \int_{surf} da\, \sigma_b = 0 \,,$$

Proof:

$$-\int_{vol} d^3r\, \vec{\nabla}\cdot\vec{P} + \int_{surf} da\, \hat{n}\cdot\vec{P}$$

$$-\int da\, \hat{n}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P} = 0$$

by divergence theorem

Electric Displacement $\vec{D}(\vec{r})$

inside the dielectric $\rho_b = -\vec{\nabla}\cdot\vec{P}$ is the "bound charge"

there may also be "free charge" ρ_f that is added to the system from some external source. The total charge is therefore $\rho = \rho_f + \rho_b$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f+\rho_b}{\varepsilon_0} = \frac{\rho_f}{\varepsilon_0} - \frac{\vec{\nabla}\cdot\vec{P}}{\varepsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot\left[\varepsilon_0\vec{E} + \vec{P}\right] = \rho_f$$

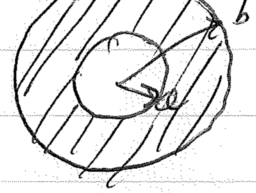
Displacement field $\vec{D}$ defined by $\boxed{\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}}$

and so $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$ Gauss' Law for dielectrics

$$\Rightarrow \oint_S \vec{D}\cdot d\vec{a} = Q_f \text{ enclosed}$$

We can use the above Gauss law for $\vec{D}$ to solve for $\vec{D}$ when we have a problem with sufficient symmetry to enable us to compute $\oint_S \vec{D}\cdot d\vec{a}$.

prob 4.15



Thick dielectric spherical shell of inner radius a outer radius b,
Frozen in polarization density
$$\vec{P} = \frac{k}{r}\hat{r}$$

free charge $= 0$.

a) Find all the bound charge and compute the resulting $\vec{E}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right)$$

$$= -\frac{k}{r^2}\frac{\partial}{\partial r}(r) = \frac{-k}{r^2}$$

at radius a: $\sigma_b = -\hat{r}\cdot\vec{P}(a) = -\frac{k}{a} \equiv \sigma$ outward normal is $\hat{n} = -\hat{r}$

at radius b: $\sigma_b = \hat{r}\cdot\vec{P}(b) = \frac{k}{b} \equiv \sigma'$ outward normal is $\hat{n} = +\hat{r}$

By the spherical symmetry of the problem we know $\vec{E}(\vec{r})$ must have the form $E(r)\hat{r}$.

Taking a Gaussian surface that is a sphere of radius r then gives

$$\oint \vec{E}\cdot d\vec{a} = 4\pi r^2 E(r) = \frac{Q_{encl}}{\varepsilon_0}$$

$$E(r) = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2}$$

Now we need to compute Q_{encl} as function of r.

For $r < a$, $Q_{encl} = 0$

For $a < r < b$

$$Q_{encl} = 4\pi a^2 \sigma + 4\pi \int_a^r dr\, r^2 \rho_b(r)$$

$\uparrow$ surface charge on inner surface

volume integral in spherical coords

$$= -4\pi k a + 4\pi \int_a^r dr\, r^2 \left(\frac{-k}{r^2} \right)$$

$$= -4\pi k a - 4\pi k (r-a) = -4\pi k r$$

For $r > b$

$$Q_{encl} = 4\pi a^2 \sigma + 4\pi b^2 \sigma' + 4\pi \int_a^b dr\, r^2 \rho_b(r)$$

$$= -4\pi k a + 4\pi k b - 4\pi k (b-a) = 0$$

$$Q_{encl} = \begin{cases} 0 & r < a \\ -4\pi k r & a < r < b \\ 0 & r > b \end{cases}$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} 0 & r < a \text{ or } r > b \\ \dfrac{-4\pi k r}{4\pi \varepsilon_0 r^2} \hat{r} & a < r < b \end{cases}$$

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < a \text{ or } r > b \\ \dfrac{-k \hat{r}}{\varepsilon_0 r} & a < r < b \end{cases}$$

b) Easier to solve using Gauss' Law for $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

by symmetry $\vec{D} = D(r)\hat{r}$

$$\oint \vec{D} \cdot d\vec{a} = Q^{free}_{enclosed}$$

$$4\pi r^2 D(r) = 0$$

$$\Rightarrow \vec{D}(\vec{r}) = 0 \Rightarrow \vec{E} = \frac{-\vec{P}}{\varepsilon_0} = \begin{cases} \dfrac{-k\,\hat{r}}{\varepsilon_0 r} & a < r < b \\ 0 & r < a \text{ or } r > b \text{ where } \vec{P} = 0 \end{cases}$$

same as in part (a)!