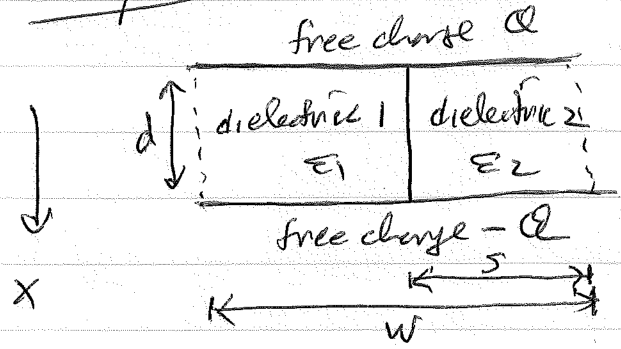
example



space between two conducting plates is filled with two dielectrics as shown

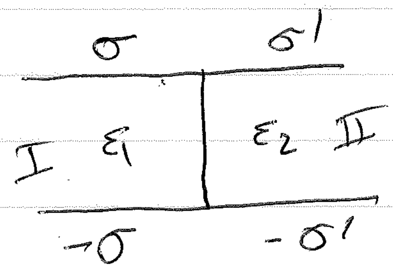
free charge +Q on top plate and −Q on bottom plate.

We know the charge Q cannot be spread out uniformly over the plates. If it was, with $\sigma_0 = Q/A$ then the electric field in dielectric 1 would be $\vec{E}_1 = \frac{\sigma_0}{\varepsilon_1}\hat{x}$ and in dielectric 2 it would be $\vec{E}_2 = \frac{\sigma_0}{\varepsilon_2}\hat{x}$

ie $\vec{E}_1 \neq \vec{E}_2$. But we must have

$-\int_0^d E \cdot dx = \Delta V$ the same on both sides (ie in both dielectrics 1 and 2) since the conducting plates must be at a constant potential

$\Rightarrow$ Q is not uniformly distributed over the plates



where $\sigma \neq \sigma'$ but

$$A\left[\sigma\left(\frac{w-s}{w}\right) + \sigma'\left(\frac{s}{w}\right)\right] = Q$$

in region I: $\vec{D} = \sigma\hat{x} \Rightarrow \vec{E}_1 = \frac{\sigma}{\varepsilon_1}\hat{x}$

in region II: $\vec{D} = \sigma'\hat{x} \Rightarrow \vec{E}_2 = \frac{\sigma'}{\varepsilon_2}\hat{x}$

Now we must have $\vec{E}_1 = \vec{E}_2 \Rightarrow \dfrac{\sigma}{\varepsilon_1} = \dfrac{\sigma'}{\varepsilon_2}$

$$\sigma' = \left(\frac{\varepsilon_2}{\varepsilon_1}\right)\sigma = \left(\frac{\kappa_2}{\kappa_1}\right)\sigma$$

Now we can find σ and σ'

we had $A\left[\sigma\left(\frac{w-s}{w}\right) + \sigma'\left(\frac{s}{w}\right)\right] = Q$

$$\sigma\left(\frac{w-s}{w}\right) + \frac{\kappa_2}{\kappa_1}\sigma\left(\frac{s}{w}\right) = \frac{Q}{A}$$

$$\sigma\left[\frac{w-s}{w} + \frac{\kappa_2}{\kappa_1}\frac{s}{w}\right] = \frac{Q}{A}$$

$$\sigma = \frac{Qw}{A}\frac{1}{\left[w-s+\frac{\kappa_2}{\kappa_1}s\right]}$$

$$= \frac{Qw}{A}\frac{1}{w+\left(\frac{\kappa_2}{\kappa_1}-1\right)s}$$

$$\sigma' = \frac{\kappa_2}{\kappa_1}\sigma = \frac{Qw}{A}\frac{\kappa_2}{\kappa_1}\frac{1}{w+\left(\frac{\kappa_2}{\kappa_1}-1\right)s}$$

We can now find the capacitance of this geometry

$$C = \frac{Q}{\Delta V} \qquad \Delta V = E d = \frac{\sigma d}{\varepsilon_1}$$

$$C = \frac{Q\,\varepsilon_1}{\sigma d} = \frac{Q\,\varepsilon_1}{d}\,\frac{A\left(w+\left(\frac{\kappa_2}{\kappa_1}-1\right)s\right)}{Qw}$$

$$C = \frac{A}{d}\varepsilon_1 \left(\frac{w + (\frac{\kappa_2}{\kappa_1} - 1)s}{w} \right)$$

$$= \frac{A}{d}\varepsilon_1 \left(1 - \frac{s}{w}\right) + \frac{A}{d}\varepsilon_1 \frac{\kappa_2}{\kappa_1}\frac{s}{w} \qquad \frac{\kappa_2}{\kappa_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

$$= \frac{A}{d}\varepsilon_1 \left(\frac{w-s}{w}\right) + \frac{A}{d}\varepsilon_2 \left(\frac{s}{w}\right)$$

$$= C_1 + C_2$$

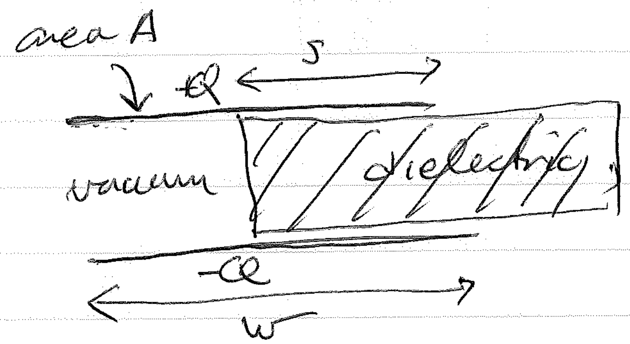
where C_1 is capacitance from dielectric 1, C_2 is capacitance from dielectric 2

$A\left(\frac{w-s}{w}\right)$ is area covered by ε_1

$A\left(\frac{s}{w}\right)$ is area covered by ε_2

Forces on dielectrics

- just like we saw that a conductor is attracted to charge due to the induced charges in the conductor, so also a dielectric is attracted to free charge due to the bound charge created by the polarization of the dielectric



dielectric inserted a distance s into the gap of a parallel plate capacitor of width w. Charges +Q on top plate, -Q on bottom plate.

we get the capacitance of this configuration from the last problem

$$C = C_{vac}\left(\frac{w-s}{s}\right) + C_{dielec}\left(\frac{s}{w}\right)$$

$$= \frac{A\varepsilon_0}{d}\left(\frac{w-s}{w}\right) + \frac{A\varepsilon}{d}\left(\frac{s}{w}\right)$$

$$= \frac{A\varepsilon_0}{dw}\left(w - s + \kappa s\right) \qquad \kappa - 1 = \chi_e$$

$$= \frac{A\varepsilon_0}{dw}\left(w + \chi_e s\right)$$

keep Q fixed, vary s

change in energy is $\frac{dW}{d\vec{s}} = \vec{F} = -\vec{F}_e$

change in work we do to vary s — force we exert — electrostatic force on dielectric

$$F_e = -\frac{dW}{ds}\bigg|_{\text{at constant } Q}$$

use $W = \frac{1}{2}\frac{Q^2}{C}$

$$F_e = -\frac{1}{2} Q^2 \frac{d}{ds}\left(\frac{1}{C}\right) = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{ds}$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{A\varepsilon_0}{dW}\chi_e$$

use $C = \frac{Q}{\Delta V}$ so

$\Delta V = \frac{Q}{C}$

$$= \frac{1}{2}(\Delta V)^2 \varepsilon_0 \chi_e \frac{A}{dW}$$

ΔV is voltage drop between the charged plates

$$F_e = \frac{-dW}{ds} > 0$$

$\Rightarrow$ as s increases, <u>W decreases</u>

we lower the energy by putting more dielectric inbetween the plates.

$\Rightarrow$ F_e pulls dielectric into the plates

We could instead have done the calculation keeping the potential drop ΔV constant, rather than keeping Q constant. But as s changes for constant ΔV, the total charge Q on the plates must change since $Q = C\Delta V$ and C changes. To keep V constant, the battery must do work by adding the necessary ΔQ as s varies.

Note: $\varepsilon_0 \chi_e = \varepsilon - \varepsilon_0$ so we can write

$$F_e = \frac{1}{2}\frac{(\Delta V)^2}{W}(C - C_0) \quad \text{where } C = \frac{A\varepsilon}{d},\ C_0 = \frac{A\varepsilon_0}{d}$$

$$F_e = -\frac{dW}{ds} + \Delta V \frac{dQ}{ds}$$

electrostatic energy

work done by the battery

use $W = \frac{1}{2}(\Delta V)^2 C$ $\qquad$ $C = \frac{Q}{\Delta V}$

$$F_e = -\frac{1}{2}(\Delta V)^2 \frac{dC}{ds} + \Delta V \frac{d}{ds}(\Delta V C)$$

$$= -\frac{1}{2}(\Delta V)^2 \frac{dC}{ds} + (\Delta V)^2 \frac{dC}{ds}$$

$$= \frac{1}{2}(\Delta V)^2 \frac{dC}{ds}$$

$$= \frac{1}{2}(\Delta V)^2 \frac{A \varepsilon_0 \chi_e}{dw}$$

same result as before so force is the same whether we keep charge Q, or potential drop ΔV, constant

4.41

Susceptibility and Atomic Polarizability

$\vec{p} = \alpha \vec{E}_{loc}$ where $\vec{E}_{loc}$ is the local electric field acting on an atom that polarizes it. $\vec{E}_{loc}$ is due to all sources other than the charge of the atom itself.

$\vec{P} = N\vec{p} = N\alpha \vec{E}_{loc}$ $\quad N$ = density of atoms

If $\vec{E}_{loc} = \vec{E}$, the average electric field, then we would have

$$\vec{P} = N\vec{p} = N\alpha\vec{E} = \varepsilon_0 \chi_e \vec{E}$$

$$\Rightarrow \chi_e = \frac{N\alpha}{\varepsilon_0}$$ relates macroscopic χ_e to microscopic α

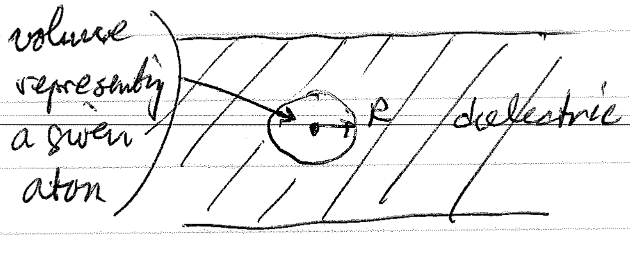
<u>BUT</u> $\vec{E} \neq \vec{E}_{loc}$ as $\vec{E}$ includes the electric field produced by the atom itself.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field due to the atom,

Let us take the volume per atom to be a sphere of radius R such that $\frac{4}{3}\pi R^3 = 1/N$

Let us say that on average this sphere, representing the portion of the system belonging to a given atom, is uniformly polarized with polarization $\vec{P}$.



Then the average field from the atom inside this sphere would be

$$\vec{E}_{atom} = \frac{-\vec{P}}{3\varepsilon_0}$$ field inside a uniformly polarized sphere

Now $\frac{4}{3}\pi R^3 \vec{\mathbb{P}} = \vec{p}$ the total dipole moment on the sphere
= total dipole moment on atom

$$\vec{p} = \alpha \vec{E}_{loc}$$

$$\Rightarrow \vec{\mathbb{P}} = \frac{3\alpha \vec{E}_{loc}}{4\pi R^3}$$

average field
$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom} = \vec{E}_{loc} - \frac{\vec{\mathbb{P}}}{3\varepsilon_0} = \vec{E}_{loc} - \frac{\alpha \vec{E}_{loc}}{4\pi\varepsilon_0 R^3}$$

$$\vec{E} = \left(1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}\right)\vec{E}_{loc}$$

$$\vec{\mathbb{P}} = N\vec{p} = N\alpha\vec{E}_{loc} = \frac{N\alpha}{1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}}\vec{E}$$

Now $\frac{4}{3}\pi R^3 = 1/N$ so

$$\vec{\mathbb{P}} = \frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_0}}\vec{E} \Rightarrow \boxed{\varepsilon_0 \chi_e = \frac{N\alpha}{1 - \frac{N\alpha}{3\varepsilon_0}}}$$

If density N is sufficiently small, we can regard the denominator as unity and regain $\varepsilon_0 \chi_e = N\alpha$ that we found when we assumed $\vec{E} = \vec{E}_{loc}$. The above thus gives the correction to this small N limit.

$$\chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0}$$

If a is the radius of the actual atom ($a \sim$ Å) then we can define

$$f = \frac{\frac{4}{3}\pi a^3}{\frac{4}{3}\pi R^3} = \frac{4}{3}\pi a^3 N$$

f is just the fraction of the space of the dielectric that is occupied by the atoms (the rest is just the empty space between atoms)

$$\chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0}$$

If we use our simple model of atomic polarizability that we discussed at the start of our section on dielectrics, then we have

$$\alpha = 4\pi\varepsilon_0 a^3$$

$$\Rightarrow \quad \chi_e = \frac{4\pi a^3 N}{1 - \frac{4}{3}\pi a^3 N} = \boxed{\frac{3f}{1-f} = \chi_e}$$ Clausius-Mossotti equation

In terms of the dielectric constant $\kappa = 1 + \chi_e$ we can write

$$\chi_e = \frac{N\alpha/\varepsilon_0}{1 - \frac{N\alpha/\varepsilon_0}{3}} \Rightarrow \chi_e - \frac{N\alpha}{3\varepsilon_0}\chi_e = \frac{N\alpha}{\varepsilon_0}$$

$$\Rightarrow \chi_e = \frac{N}{\varepsilon_0}\left(1 + \frac{\chi_e}{3}\right)\alpha$$

$$\Rightarrow \alpha = \frac{\varepsilon_0}{N}\frac{\chi_e}{1 + \frac{\chi_e}{3}} = \frac{3\varepsilon_0}{N}\frac{\chi_e}{3 + \chi_e}$$

$$\boxed{\alpha = \frac{3\varepsilon_0}{N}\left(\frac{\kappa - 1}{\kappa + 2}\right)}$$ ← Lorentz-Lorenz equation

Relates microscopic parameter α to macroscopic parameter κ.