Magnetostatics

I) Charge conservation

ρ charge density

$$\boxed{\vec{j} = \rho\vec{v}} = \text{charge current density}$$

$\hat{j}\cdot\hat{n}\, da$ is total charge per unit time flowing through area da where $\hat{n}$ is normal to da

$$\frac{dQ_{encl}}{dt} = -\oint_S \vec{j}\cdot d\vec{a}$$

← flux of current through surface S

change in total charge enclosed by S

$$\Rightarrow \frac{d}{dt}\int_V d^3r\, \rho = -\int_V d^3r\, \vec{\nabla}\cdot\vec{j} \Rightarrow \boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}}$$ charge conservation

since integrals equal for any V

If we want a "steady state" or "static" situation, we can only consider currents $\vec{j}$ that satisfy $\vec{\nabla}\cdot\vec{j} = 0$ so that $\partial\rho/\partial t = 0$.

Magnetostatics deals with effects of moving charges, such that the current is always divergenceless

$$\boxed{\vec{\nabla}\cdot\vec{j} = 0}$$

Simplest example: constant uniform current I flowing down a wire. $I = \vec{j}\cdot d\vec{a}$ where $d\vec{a}$ is cross section area of wire, points along tangent to wire

~~Static steady state currents produce magnetic fields $\vec{B}$~~

units of current I is amps = coulomb/sec

units of current density $\vec{j}$ is $\frac{\text{amps}}{m^2}$

$\vec{J}$ is a "volume" current density.

Can also have "sheet" or "surface" current density

$\vec{K} = \sigma \vec{v}$ when surface charge σ on 2-d surface moves

Also have "line" current density

$\vec{I} = \lambda \vec{v}$ when line charge λ on 1-d line moves
- like current flowing in a wire.

* Steady state currents produce magnetic fields $\vec{B}(\vec{r})$

II) Biot-Savart Law (Analog to Coulombs Law) ("steady state")

The magnetic field at point $\vec{r}$, due to a divergenceless current flow is:

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{vol} d^3r' \; \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

for volume current density $\vec{J}(\vec{r}')$

$$= \frac{\mu_0}{4\pi} \int_{surface} da' \; \vec{K}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

for surface current density $\vec{K}(\vec{r}')$, $\vec{r}'$ on surface

$$= \frac{\mu_0}{4\pi} \int_{line} dl' \; \vec{I}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

for line current density $\vec{I}(\vec{r}')$, $\vec{r}'$ on line

$\mu_0 = 4\pi \times 10^{-7}$ nt/amp^2 fundamental constant of magnetostatics

"permeability" of free space

* Magnetic fields produces forces on moving charges

III) $\vec{F}_{mag} = q\,\vec{v}\times\vec{B}(\vec{r})$ Lorentz force

force on charge q, at position $\vec{r}$, moving with velocity $\vec{v}$, due to magnetic field $\vec{B}(\vec{r})$

total electromagnetic force is $\vec{F}_{em} = q\,\vec{E}(\vec{r}) + q\,\vec{v}\times\vec{B}(\vec{r})$

magnetic force on a current density

$$\vec{F}_{mag} = \int_{vol} d^3r\ \rho(\vec{r})\,\vec{v}(\vec{r})\times\vec{B}(\vec{r})$$

$$= \int_{vol} d^3r\ \vec{J}(\vec{r})\times\vec{B}(\vec{r})$$ for volume current density

$$= \int_{surface} da\ \vec{K}(\vec{r})\times\vec{B}(\vec{r})$$ for surface current density $\vec{r}$ on surface

$$= \int_{line} dl\ \vec{I}(\vec{r})\times\vec{B}(\vec{r})$$ for line current density $\vec{r}$ on line

For line current, can rewrite:

$$\vec{I} = I\,\hat{t}$$ unit tangent vector to line

$$\vec{F}_{mag} = I\int dl\,\hat{t}\times\vec{B}(\vec{r}) = I\int(d\vec{l}\times\vec{B})$$

where $dl\,\hat{t} \equiv d\vec{l}$ as in vector line integrals magnetic forces can do no work

$$W_{mag} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}_{mag} \cdot d\vec{l}$$

work done by magnetic field in moving a charge from $\vec{r}_a$ to $\vec{r}_b$

$$= q \int_{\vec{r}_a}^{\vec{r}_b} (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= q \int_{t_a}^{t_b} (\vec{v} \times \vec{B}) \cdot \frac{d\vec{l}}{dt}\, dt$$

$$= q \int_{t_a}^{t_b} \left[(\vec{v} \times \vec{B}) \cdot \vec{v} \right] dt$$

integrate along trajectory $\vec{r}(t)$ that charge takes in going from $\vec{r}_a$ to $\vec{r}_b$

$\vec{r}(t_a) = \vec{r}_a$, $\vec{r}(t_b) = \vec{r}_b$

$\vec{v}(t) = \frac{d\vec{r}}{dt}$

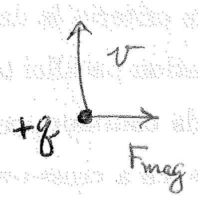
$$= 0$$

since $(\vec{v} \times \vec{B}) \cdot \vec{v}$

$= (\vec{v} \times \vec{v}) \cdot \vec{B}$

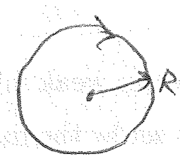
$= 0$

Cyclotron Motion



$F_{mag} = q\,\vec{v} \times \vec{B}$

$\Rightarrow$ orbital motion



+q moves in clockwise direction when $\vec{B}$ out of page

centrip accel $\quad a_c = \dfrac{v^2}{R} = \dfrac{F_{mag}}{m} = \dfrac{qvB}{m}$

$$R = \frac{v^2 m}{q v B} = \frac{vm}{qB}$$

$$\boxed{R = \frac{vm}{qB}}$$

angular velocity is: $\dfrac{v}{R} = \dfrac{qB}{m} \equiv \omega_c$ cyclotron frequency

$\vec{\nabla}\cdot\vec{B} = ?$
$\vec{\nabla}\times\vec{B} = ?$

IV) Maxwells Equations for Magnetostatics:

Biot-Savart law: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \vec{j}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \vec{\nabla}\cdot\left(\vec{j}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)$$

use vector identity

$$\vec{\nabla}\cdot(\underbrace{\vec{v}}_{\text{indep of }\vec{r}}\times\underbrace{\vec{w}(\vec{r})}_{\text{depends on }\vec{r}}) = -\vec{v}\cdot(\vec{\nabla}\times\vec{w}(\vec{r}))$$

curl of a radial function vanishes

$$\Rightarrow \vec{\nabla}\cdot\vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi}\int d^3r'\ \vec{j}(\vec{r}\,')\cdot\underbrace{\vec{\nabla}\times\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right)}_{=0}$$

$=0$ - we saw this before when we used coulombs law to show that $\vec{\nabla}\times\vec{E}=0$ in electrostatics

Gauss Law for Magnetic fields

$$\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0}$$

"no magnetic monopoles"

$$\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \vec{\nabla}\times\left(\vec{j}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)$$

use vector identity

$$\vec{\nabla}\times(\underbrace{\vec{v}}_{\text{indep of }\vec{r}}\times\underbrace{\vec{w}(\vec{r})}_{\text{depends on }\vec{r}}) = -(\vec{v}\cdot\vec{\nabla})\vec{w} + \vec{v}(\vec{\nabla}\cdot\vec{w})$$

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \left\{ \vec{J}(\vec{r}\,') \, \vec{\nabla} \cdot \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right) - \left(\vec{J}(\vec{r}\,') \cdot \vec{\nabla} \right) \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right) \right\}$$

1st term: $\vec{\nabla} \cdot \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right) = 4\pi \, \delta^3(\vec{r}-\vec{r}\,')$

$\Rightarrow$ 1st Term gives $\frac{\mu_0}{4\pi} \int d^3r' \, \vec{J}(\vec{r}\,') \, 4\pi \delta^3(\vec{r}-\vec{r}\,') = \mu_0 \vec{J}(\vec{r})$

2nd term is $\frac{\mu_0}{4\pi} \int d^3r' (-) \sum_{i=1}^{3} J_i(\vec{r}\,') \frac{\partial}{\partial x_i} \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right)$

use $\frac{\partial}{\partial x_i} \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right) = -\frac{\partial}{\partial x_i'} \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right)$

$$= \frac{\mu_0}{4\pi} \int d^3r' \sum_{i=1}^{3} J_i(\vec{r}\,') \frac{\partial}{\partial x_i'} \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right)$$

integrate by parts: boundary term vanishes as take surface $S \to \infty$ if current $\vec{J}$ is localized

$$= -\frac{\mu_0}{4\pi} \int d^3r' \underbrace{\sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i'} J_i(\vec{r}\,') \right)}_{\nabla' \cdot \vec{J}(\vec{r}\,') = 0} \left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \right)$$

2nd term $= 0$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})}$$ Ampere's Law

$$\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$$
$$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$$
} Maxwell's equ for magnetostatics

Ampere's law in integral form:

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{B}(\vec{r})) = \mu_0 \int_S d\vec{a}\cdot\vec{J}(\vec{r})$$ S is any open surface

Stokes law $\quad \parallel \qquad\qquad \parallel$

$$\boxed{\oint_\Gamma d\vec{l}\cdot\vec{B} = \mu_0 I_{encl}}$$

Γ boundary curve of S

I_{encl}: total flux of $\vec{J}$ through surface S = "current enclosed" by bounding loop Γ

V) Magnetic Vector Potential

Since $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$ can always find $\vec{A}(\vec{r})$ such that

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$$ $\vec{A}$ is the "vector potential" for $\vec{B}$

general result of vector calculus (see Theorem 2 sec 1.6.2)

Note: $\vec{A}$ is not unique: if $\vec{B} = \vec{\nabla}\times\vec{A}$

and $\vec{A}' \equiv \vec{A} + \vec{\nabla}\lambda$ where $\lambda(\vec{r})$ is any scalar function

Then $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\lambda)$

$= \vec{B} + 0$ as $\vec{\nabla}\times\vec{\nabla}\lambda = 0$ always

$\Rightarrow \vec{A}'$ can also be used as a vector potential for $\vec{B}$.

Note:

This is similar to case for electrostatic potential:

$\vec{E} = -\vec{\nabla} V \Rightarrow$ V is not unique - we can always add any arbitrary constant to V and still not change $\vec{\nabla} V$.

Here $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{A}$ is not unique - we can always add any arbitrary $\vec{\nabla}\lambda$ to $\vec{A}$ and not change $\vec{\nabla} \times \vec{A}$

Amperes Law in terms of vector potential:

$$\underbrace{\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})}_{\text{definition of } \vec{A}} \underbrace{= -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})}_{\text{vector identity}} \underbrace{= \mu_0 \vec{J}}_{\text{amperes law}}$$

we can always use the non uniqueness of $\vec{A}$ to choose an $\vec{A}$ that satisfies $\vec{\nabla} \cdot \vec{A} = 0$, called "Coulomb gauge".

proof: Suppose we have an $\vec{A}'$ such that $\vec{\nabla} \times \vec{A}' = \vec{B}$ but $\vec{\nabla} \cdot \vec{A}' = C \neq 0$

1) Find a λ such that $-\nabla^2 \lambda = C(\vec{r})$.

This is Poisson's equ, so we know that there is always a solution $\lambda(\vec{r})$.

2) Then construct $\vec{A}(\vec{r}) \equiv \vec{A}'(\vec{r}) + \vec{\nabla}\lambda(\vec{r})$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}' + \vec{\nabla} \times \vec{\nabla}\lambda = \vec{\nabla} \times \vec{A}' + 0 = \vec{B}$$
$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}' + \vec{\nabla} \cdot (\vec{\nabla}\lambda) = C + \nabla^2 \lambda = C - C = 0$$

So $\vec{A}$ is a vector potential for $\vec{B}$, and $\vec{\nabla} \cdot \vec{A} = 0$

For magnetostatics, it is usually convenient to always work with vector potential that satisfies $\vec{\nabla}\cdot\vec{A} = 0$.

Then Amperes law becomes

$$\boxed{-\nabla^2\vec{A} = \mu_0\vec{J}}$$

Solution is
for localized current

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

vector equation

just like we solved $-\nabla^2 V = \rho/\varepsilon_0$ in electrostatics

for surface current density: $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S da' \, \frac{\vec{K}(\vec{r}')}{|\vec{r}-\vec{r}'|}$ $\quad \vec{r}'$ on surface S

for line current density:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_P dl' \, \frac{\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|} \qquad \vec{r}' \text{ on line } P$$