Magnetostatics Summary - Comparison with Electrostatics

I) charge conservation $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ for magnetostatics

Magnetostatics	Electrostatics
II) Biot-Savart law	Coulomb's law
$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \vec{J}(\vec{r}\,') \times \frac{(\vec{r}-\vec{r}\,')}{\lvert\vec{r}-\vec{r}\,'\rvert^3}$	$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\, \rho(\vec{r}\,') \frac{(\vec{r}-\vec{r}\,')}{\lvert\vec{r}-\vec{r}\,'\rvert^3}$
III) Lorentz force	Coulomb force
$\vec{F}_{mag} = q\,\vec{v}\times\vec{B}$	$\vec{F}_{elec} = q\vec{E}$
IV) Maxwell equations (from Biot-Savart)	Max Eq (from Coulomb)
$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ Gauss Law
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ Ampere law	$\vec{\nabla}\times\vec{E} = 0$
$\oint_{\Gamma} d\vec{l}\cdot\vec{B} = \mu_0 I_{encl}$ integral form	$\oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{encl}}{\varepsilon_0}$ integral form
V) Magnetic vector potential	Electrostatic scalar potential
$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$	$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$
$\vec{A}$ not unique: $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ OK also	V not unique: $V' = V + V_0$ OK also
if choose λ such that $\vec{\nabla}\cdot\vec{A} = 0$, then	
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow -\nabla^2\vec{A} = \mu_0\vec{J}$	$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow -\nabla^2 V = \rho/\varepsilon_0$
if $\vec{J}\to 0$ as $\vec{r}\to\infty$, then	if choose V_0 such that $V(\vec{r})\to 0$ as $\vec{r}\to\infty$, and $\rho\to 0$ as $r\to\infty$ then
$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{J}(\vec{r}\,')}{\lvert\vec{r}-\vec{r}\,'\rvert}$	$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\, \frac{\rho(\vec{r}\,')}{\lvert\vec{r}-\vec{r}\,'\rvert}$

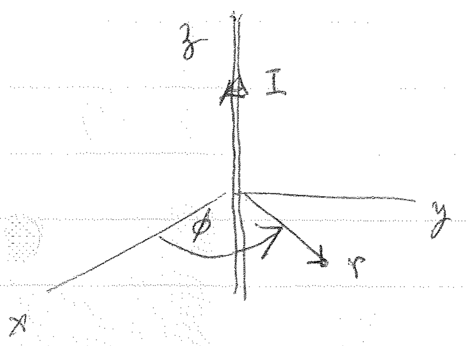
Biot - Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\ \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

for a line current: $\displaystyle \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int dl'\ \vec{I} \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

ex: straight wire along z axis with current I

What is field at pt $\vec{r}$? Consider pt $\vec{r} = (r, \phi, z=0)$ in cylindrical coords. By translation symmetry, $\vec{B}(\vec{r})$ is indep of z coordinate, so it is enough to find $\vec{B}$ at $z=0$.

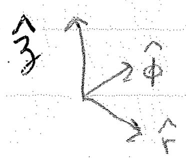


$\vec{I} = I\hat{z}$, $\vec{r} = r\,\hat{r}$, $\vec{r}' = z\hat{z}$

$\hat{r}$: cylindrical radial basis vector

$\vec{r}'$: line is along z axis

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} dz\ \hat{z} \times \frac{(r\hat{r} - z\hat{z})}{([r\hat{r} - z\hat{z}]^2)^{3/2}}$$

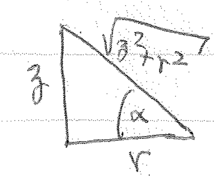


$(r\hat{r} - z\hat{z})^2 = r^2 + z^2$

$\hat{z} \times (r\hat{r} - z\hat{z}) = r\hat{\phi}$ as $\hat{z}\times\hat{r} = \hat{\phi}$, $\hat{z}\times\hat{z} = 0$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I r\hat{\phi} \int_{-\infty}^{\infty} dz\ \frac{1}{(r^2+z^2)^{3/2}}$$

change integration variable

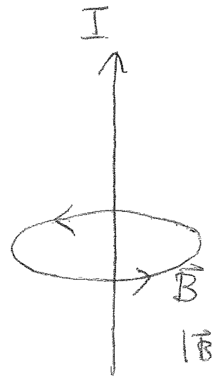


$\sqrt{r^2+z^2} = \dfrac{r}{\cos\alpha}$, $z = r\tan\alpha$

$dz = r\,\dfrac{d\tan\alpha}{d\alpha}\,d\alpha = \dfrac{r}{\cos^2\alpha}\,d\alpha$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I r\hat{\phi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \left(\frac{r}{\cos^2\alpha}\right)\left(\frac{\cos\alpha}{r}\right)^3$$

$$= \frac{\mu_0}{4\pi}\frac{I\hat{\phi}}{r}\underbrace{\int_{-\pi/2}^{\pi/2} d\alpha\ \cos\alpha}_{2} = \boxed{\frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi} = \vec{B}(\vec{r})}$$

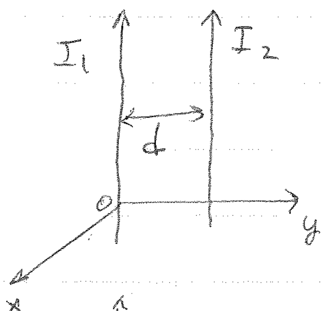


$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{I}{r}\hat{\phi}}$$

$\vec{B}$ is in $\hat{\phi}$ direction

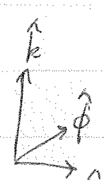
$|\vec{B}|$ decays as $\frac{1}{r}$ as distance from wire increases

Force of attraction between two parallel wires



force on I_2 due to magnetic field of ~~wire~~ I_1

$\vec{B}_1(d\hat{y}) = \frac{\mu_0}{2\pi}\frac{I_1}{d}\hat{\phi}$ field from I_1, at position of I_2



Force on I_2 is $\vec{F}_{mag} = \int dz(\vec{I}_2 \times \vec{B}_1) = L I_2 \frac{\mu_0}{2\pi}\frac{I_1}{d}\hat{z}\times\hat{\phi}$

$\vec{B}_1$ is const along wire 2

$= \frac{\mu_0 L}{2\pi d} I_1 I_2 (-\hat{r})$ as $\hat{z}\times\hat{\phi} = -\hat{r}$

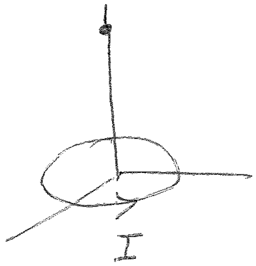
$\vec{F}_{mag} = \frac{-\mu_0 L}{2\pi d} I_1 I_2 \hat{r}$

radial vector from 1 to 2

if I_1 and I_2 are in same direction, so $I_1 I_2 > 0$, force is <u>attractive</u>.

if I_1 and I_2 in opposite directions, so $I_1 I_2 < 0$, force is <u>repulsive</u>.

Circular loop   find $\vec{B}$ on z axis



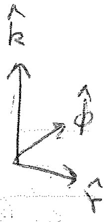
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_\Gamma dl\; \vec{I} \times \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

here $\vec{r} = z\hat{z}$

$\vec{r}\,' = R\hat{r}$ cylindrical $\hat{r}$

$\vec{I} = I\hat{\phi}$ $\int dl = \int_0^{2\pi} d\phi\, R$ differential dl = arclength

$$\vec{B}(z\hat{z}) = \frac{\mu_0}{4\pi}\int_0^{2\pi} d\phi\, R\, I\, \hat{\phi} \times \frac{(z\hat{z} - R\hat{r})}{|z\hat{z} - R\hat{r}|^3}$$



$\hat{\phi} \times \hat{z} = \hat{r}$

$\hat{\phi} \times \hat{r} = -\hat{z}$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi\, R\, \frac{(z\hat{r} + R\hat{z})}{(z^2+R^2)^{3/2}}$$

$\hat{r} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$

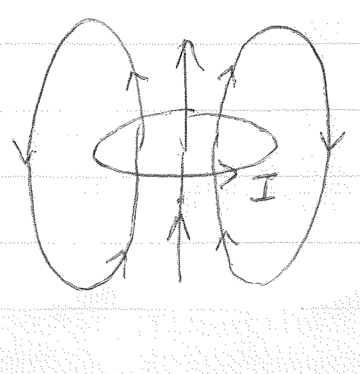
$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\phi\, R\, \frac{(z\cos\phi\,\hat{\imath} + z\sin\phi\,\hat{\jmath} + R\hat{z})}{(z^2+R^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi}\frac{R^2}{(z^2+R^2)^{3/2}}\, 2\pi$$

$\int_0^{2\pi}\cos\phi = \int_0^{2\pi}\sin\phi = 0$

$$\boxed{\vec{B}(z\hat{k}) = \frac{\mu_0 R^2 I}{2(z^2+R^2)^{3/2}}}$$

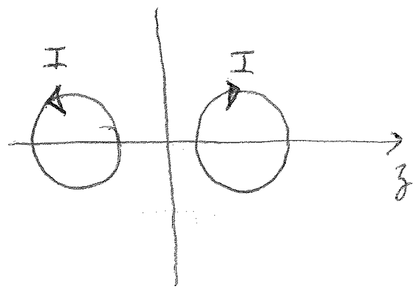
along z axis



see later that $\vec{B}$ looks like dipole field

Symmetry and Magnetic Field

Suppose the xy plane at $z=0$ is a plane of reflection symmetry for the current source



For current density $\vec{j}$, to be reflection symmetric, it should behave like the position vector upon reflection, ie:

example: two current carrying wires as shown has reflection symmetry in ~~about~~ xy plane

$j_x(x,y,z) = j_x(x,y,-z)$ sym

$j_y(x,y,z) = j_y(x,y,-z)$ sym

$j_z(x,y,z) = -j_z(x,y,-z)$ antis

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} \Rightarrow$$

$$\frac{\partial B_z(x,y,z)}{\partial y} - \frac{\partial B_y(x,y,z)}{\partial z} = \mu_0 j_x(x,y,z) = \mu_0 j_x(x,y,-z) = \frac{\partial B_z(x,y,-z)}{\partial y} - \frac{\partial B_y(x,y,-z)}{\partial(-z)}$$

$$\frac{\partial B_x(x,y,z)}{\partial z} - \frac{\partial B_z(x,y,z)}{\partial x} = \mu_0 j_y(x,y,z) = \mu_0 j_y(x,y,-z) = \frac{\partial B_x(x,y,-z)}{\partial(-z)} - \frac{\partial B_z(x,y,-z)}{\partial x}$$

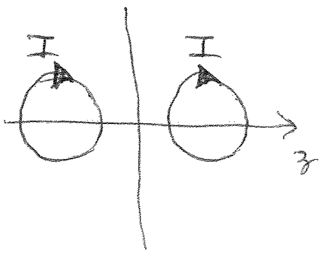
$$\frac{\partial B_y(x,y,z)}{\partial x} - \frac{\partial B_x(x,y,z)}{\partial y} = \mu_0 j_z(x,y,z) = -\mu_0 j_z(x,y,-z) = -\frac{\partial B_y(x,y,-z)}{\partial x} + \frac{\partial B_x(x,y,-z)}{\partial y}$$

comparing leftmost and right most sides of equation gives

$$\vec{j}\ \text{symmetric} \Rightarrow \begin{cases} B_x(x,y,z) = -B_x(x,y,-z) & \text{antisymmetric} \\ B_y(x,y,z) = -B_y(x,y,-z) & \text{antisymmetric} \\ B_z(x,y,z) = B_z(x,y,-z) & \text{symmetric} \end{cases}$$

$\rightarrow B_x(x,y,0) = B_y(x,y,0) = 0$ or $\vec{B}\perp$ symmetry plane at $z=$

Similarly, if $\vec{j}$ has antisymmetry with respect xy reflection plane



ie:

$$j_x(x,y,z) = -j_x(x,y,-z) \quad \text{antisym}$$
$$j_y(x,y,z) = -j_y(x,y,-z) \quad \text{antisym}$$
$$j_z(x,y,z) = j_z(x,y,-z) \quad \text{sym}$$

example antisymmetric $\vec{j}$

then:

$$B_x(x,y,z) = B_x(x,y,-z) \quad \text{symmetric}$$
$$B_y(x,y,z) = B_y(x,y,-z) \quad \text{symmetric}$$
$$B_z(x,y,z) = -B_z(x,y,-z) \quad \text{antisymmetric}$$

$\Rightarrow$ $B_z(x,y,0)=0$ or $\vec{B} \parallel$ symmetry plane at $z=0$.

Note that whichever symmetry property $\vec{j}$ has (reflection symmetric or reflection antisymmetric) $\vec{B}$ has the reverse symmetry property, ie ~~there is~~ $\vec{B}$ behaves like $\vec{j}$, but with extra $(-)$ sign!

This is different from what we found for symmetry properties of $\vec{E}$. There we found:

ρ reflection symmetric $\Rightarrow$ $\vec{E}$ reflection symmetric vector

ρ anti reflection symmetric $\Rightarrow$ $\vec{E}$ reflection antisym vector.

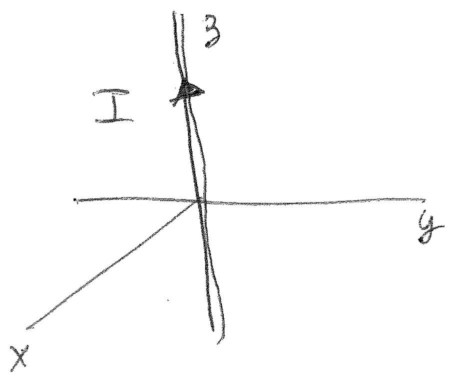
Since $\vec{E}$ has same symmetry behavior as ρ

But $\vec{B}$ has opposite symmetry behavior as $\vec{j}$

we say $\vec{E}$ transforms like a vector but $\vec{B}$ transforms as a pseudo vector

Straight wire

Reflection symmetry properties



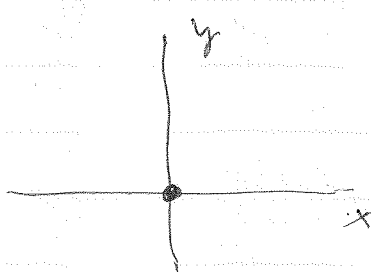
$\vec{I}$ is reflection anti symmetric about xy plane at $z = 0$

$\Rightarrow$ $\vec{B}$ is reflection symmetric about xy plane

$\Rightarrow$ $B_z(x, y, 0) = -B_z(x, y, 0) = 0$

$\vec{B}$ must lie in symmetry plane

But since $\vec{B}$ is indep of z by translational symmetry $\Rightarrow B_z(x, y, z) = 0$ for any z. $\Rightarrow$ $\vec{B}$ has no component along $\hat{z}$



$\vec{I}$ out of page

$\vec{I}$ is reflection symmetric about yz plane $\Rightarrow$ $\vec{B}$ is reflection antisymmetric

$\Rightarrow$ $B_z(0, y, z) = -B_z(0, y, z) = 0$

$B_y(0, y, z) = -B_y(0, y, z) = 0$

$\Rightarrow$ $\vec{B}$ lies $\perp$ to symmetry plane

i.e. $\vec{B}$ is along $\hat{x}$. $\Rightarrow$ no radial component.

By rotational symmetry about z axis we therefore conclude that $\vec{B}$ only points in $\hat{\phi}$ direction, only depends on r

$$\vec{B}(\vec{r}) = B(r)\,\hat{\phi}$$