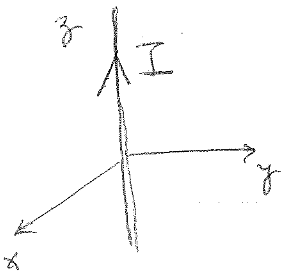
Straight wire along z axis with current I.

Solution via Maxwell's equations

$$\vec{\nabla}\cdot\vec{B}=0\ ,\quad \vec{\nabla}\times\vec{B}=\mu_0\vec{J}\ \Rightarrow\ \int_\Gamma d\vec{l}\cdot\vec{B}=\mu_0 I_{encl}$$



By symmetry, expect $\vec{B}(\vec{r})=B(r)\hat{\phi}$

$\uparrow$ cylindrical radial coord

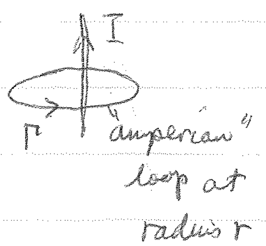
$\vec{B}$ is indep of z by translational symmetry along z axis

$\vec{B}$ must be indep of ϕ by rotation symmetry about z axis

$\Rightarrow \vec{B}$ depends only on r.

$\vec{B}$ must point in $\hat{\phi}$ direction to have correct symmetry under reflections.

<u>integral solution</u>



$$\oint_\Gamma d\vec{l}\cdot\vec{B}=\int_0^{2\pi}\underbrace{d\phi\, r\hat{\phi}}_{d\vec{l}}\cdot\vec{B}(r)=\int_0^{2\pi}d\phi\, rB(r)=2\pi r B(r)$$

$$=\mu_0 I$$

$$d\vec{l}=r\,d\phi\,\hat{\phi}$$

$$\Rightarrow\ B(r)=\frac{\mu_0}{2\pi r}I$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{2\pi r}I\hat{\phi}\quad \text{as before}$$

<u>differential solution</u>

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}=\mu_0 I\,\delta(r)$$

$\uparrow$ cylindrical coord

$\vec{J}=0$ except on z axis at $r=0$

for $\vec{B}=B(r)\hat{\phi}$

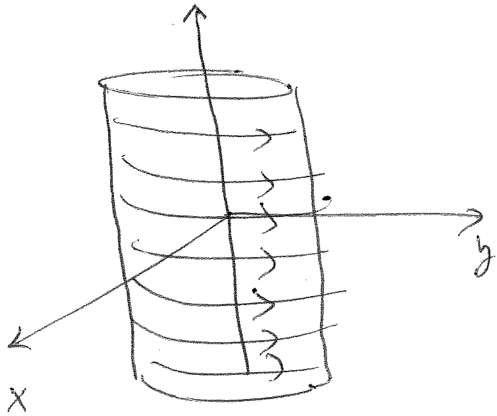
$$\vec{\nabla}\times(B\hat{\phi})=-\frac{\partial B}{\partial z}\hat{r}+\frac{1}{r}\frac{\partial}{\partial r}(rB)\quad \text{as } B_r=B_z=0$$

since $\frac{\partial B(r)}{\partial z}=0$

$$=\frac{1}{r}\frac{\partial}{\partial r}(rB)=\mu_0 I\,\delta(r)$$

$$\frac{\partial}{\partial r}(rB)=\mu_0 I\, r\,\delta(r)$$

Infinite Solenoid



surface of infinite cylinder of radius R has surface current $\vec{K} = K\hat{\phi}$.

Can produce solenoid by wrapping wire around cylinder with N turns per unit length. Then

$$K = IN$$

where I is current in wire

Symmetries: translational symmetry along z axis

$\Rightarrow \vec{B}$ indep of z

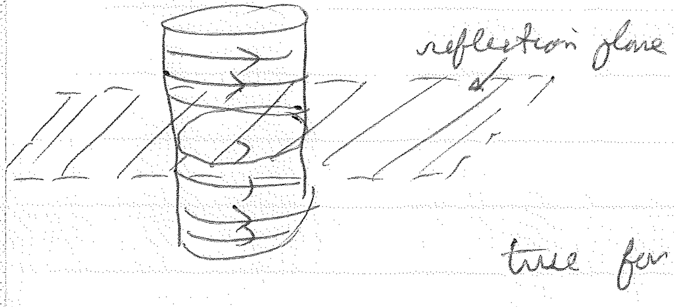
rotational symmetry about z axis

$\Rightarrow \vec{B}$ indep of ϕ

$\Rightarrow \vec{B}$ depends only on cylindrical radial coord r.

Reflect through xy plane at $z = z_0$ a constant

current is reflection symmetric



$\Rightarrow \vec{B}$ is antisymmetric

$\Rightarrow \vec{B}(x,y,z_0)$ is $\perp$ symmetry plane

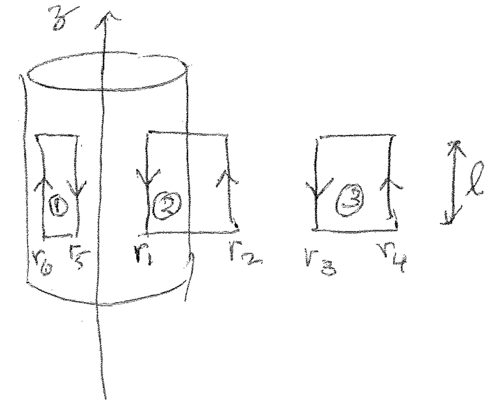
true for all z_0

$$\Rightarrow \vec{B}(x,y,z_0) = B(r)\,\hat{z}$$

so $\vec{B}$ points only in $\hat{z}$ direction

depends only on radial coord

Conclusion: $\vec{B}(\vec{r}) = B(r)\hat{z}$ ~~along~~ pts along z axis



for any loop with normal in $\hat{\phi}$ direction with inner edge at r and outer edge at r'

$$\oint_{\Gamma} \vec{B}\cdot d\vec{l} = \underbrace{B(r')\hat{z}}_{\vec{B}} \cdot \underbrace{\hat{z} l}_{d\vec{l}} - \underbrace{B(r)\hat{z}}_{\vec{B}} \cdot \underbrace{\hat{z} l}_{d\vec{l}}$$

$$= [B(r') - B(r)]l = \mu_0 I_{encl}$$

for loop ① and loop ③, $I_{encl} = 0$ as no current passes through loop! $\Rightarrow$ $B(r') = B(r)$ r, r' both outside

$B(r') = B(r)$ r, r' both inside

$\Rightarrow \vec{B}$ is constant outside and inside.

Since we expect $\vec{B} = 0$ as $r \to \infty$, infinitely far away from solenoid, $\Rightarrow \vec{B}_{outside} = 0$

for loop ②, $I_{encl} = -Kl \Rightarrow B(r') - B(r) = -\mu_0 K$

$\Rightarrow B_{outside} - B_{inside} = -\mu_0 K$

$I_{encl} = \int d\vec{a} \cdot \vec{J}$

for orientation of loop shown,

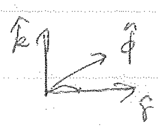
$d\vec{a} = l(r_2 - r_1)(-\hat{\phi})$

$\vec{J} = \vec{K}\,\delta(r-R)\hat{\phi}$

$\Rightarrow I_{encl} = -Kl$

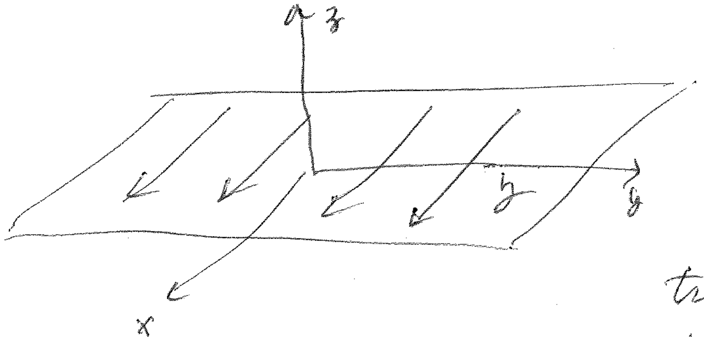
$\Rightarrow B_{inside} = \mu_0 K$

$$\vec{B}(\vec{r}) = \begin{cases} \mu_0 K \hat{z} \\ = \mu_0 N I \hat{z} & \text{inside} \\ = 0 & \text{outside} \end{cases}$$



Note: we have a discontinuous jump in $\vec{B}$ as cross a surface current. $\vec{B}_{above} - \vec{B}_{below} = -\mu_0 N I \hat{z} = -\mu_0 K \hat{z}$ will be generally true! $= \mu_0(\vec{K} \times \hat{n})$

Infinite flat uniform sheet current



$\vec{K} = K\hat{x}$

translational symmetry in x and y directions $\Rightarrow$ $\vec{B}$ indep of x and y

$\vec{B}$ depends only on z

Reflection symmetries current has reflection symmetry about xz plane at any fixed y_0

$\Rightarrow$ $\vec{B}$ is reflection antisymmetric

$\Rightarrow$ $\vec{B}(x, y_0, z) \perp$ xz plane ie. $\vec{B}$ along $\hat{y}$

true for all y_0 $\Rightarrow$ $\boxed{\vec{B}(\vec{r}) = B(z)\,\hat{y}}$

this result is consistent with: current has reflection antisymmetry about yz plane at fixed x_0

$\Rightarrow$ $\vec{B}$ is reflection symmetric

$\Rightarrow$ $\vec{B}(x_0, y, z)$ must lie in yz plane

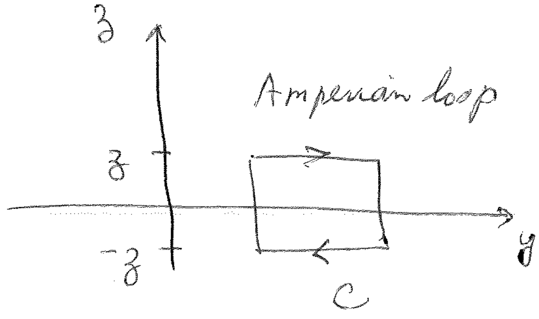
Also: current has reflection symmetry about xy plane at $z=0$

$\Rightarrow$ $\vec{B}$ is reflection antisymmetric

$\Rightarrow B_y(x, y, z) = -B_y(x, y, -z)$

for $\vec{B}(\vec{r}) = B(z)\,\hat{y}$

$$B(z) = -B(-z) \qquad \Rightarrow B(z=0) = 0$$



$$\oint_C \vec{B}\cdot d\vec{l} = [B(z) - B(-z)]\,l$$

$$= \mu_0 I_{encl} = -\mu_0 K l$$

(−) come from $I_{encl} = \int d\vec{a}\cdot\vec{J}$. Right hand rule tells us $d\vec{a}$ points <u>into</u> page, while current $\vec{J}$ points <u>out</u> of page

$$B(z) - B(-z) = 2B(z) = -\mu_0 K$$

$$B(z) = -\frac{\mu_0 K}{2} \qquad z > 0$$

$$B(z) = +\frac{\mu_0 K}{2} \qquad z < 0$$

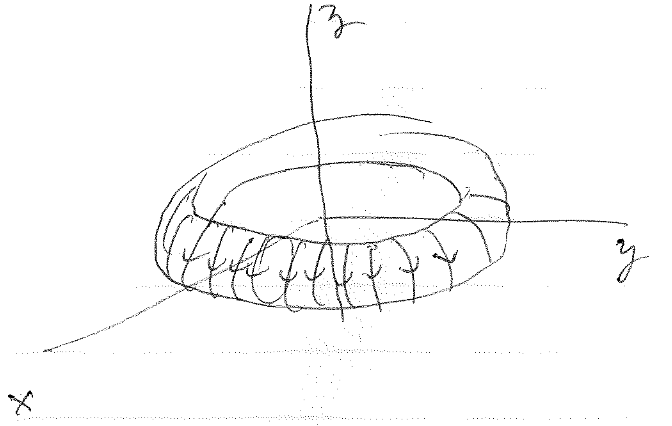
$$\vec{B}(\vec{r}) = \begin{cases} -\frac{\mu_0 K}{2}\,\hat{y} & z > 0 \\ +\frac{\mu_0 K}{2}\,\hat{y} & z < 0 \end{cases}$$

<u>As with solenoid, we have:</u>

$$\vec{B}_{above} - \vec{B}_{below} = -\frac{\mu_0 K}{2}\,\hat{y} - \frac{\mu_0 K}{2}\,\hat{y} = -\mu_0 K\,\hat{y}$$

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0(\vec{K}\times\hat{n}) \qquad \text{where } \hat{n} = \hat{z}$$

Toroidal Solenoid

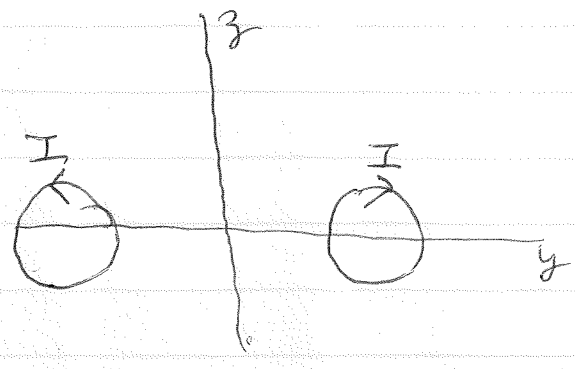


donut wraped with wire as in a solenoid.

n turns of wire around the soliaid

cross section of solenoid is any constant shape.

Reflection symmetries



cross section of current distribution in yz plane at $x=0$

Current is reflection symmetric about xz plane, at $y=0$

$\Rightarrow \vec{B}$ is reflection antisymmetri

$\Rightarrow \vec{B}(x,0,z) \perp xz$ plane

ie is along $\hat{y}$ direct

Rotation invarience about $\hat{z}$ axis then implies that $\vec{B}$ is always in $\hat{\phi}$ directi

$\Rightarrow \vec{B}(\vec{r})$ is indep of ϕ coord, and points in $\hat{\phi}$ direction

$\Rightarrow \vec{B}(\vec{r}) = B(r,z)\,\hat{\phi}$

Take Amperian loop at fixed radius r centered abou z axis at fixed z

$$\oint_C \vec{B}\cdot d\vec{l} = 2\pi r\, B(r,z) = \mu_0 I_{encl}$$

here $I_{encl} = 0$ if loop is not inside the solenoid

$I_{encl} = nI$ if loop is inside the solinoid

$$\Rightarrow \quad \vec{B}(\vec{r}) = \begin{cases} \dfrac{\mu_0 m I}{2\pi r}\,\hat{\phi} & \vec{r} \text{ inside solenoid} \\ 0 & \vec{r} \text{ outside solenoid} \end{cases}$$

Note, if $m = \overset{N\,2\pi R}{\cancel{NR}}$ with N = # turns per unit lgth

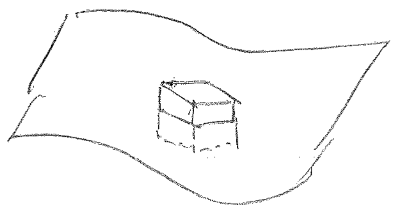
R = average radius of torous

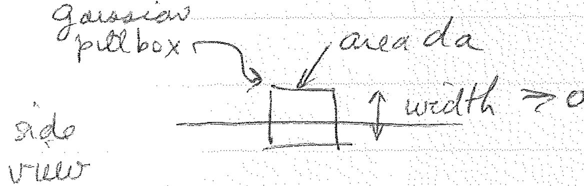
$$\vec{B}(\vec{r}) = \mu_0 N I \left(\frac{R}{r}\right) \hat{\phi}$$

similar to solenoid (straight)

Boundary Condition at sheet of surface ~~charge~~ current

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$



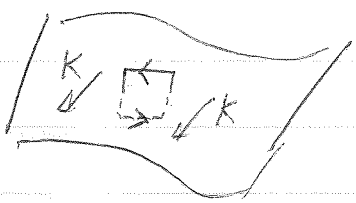


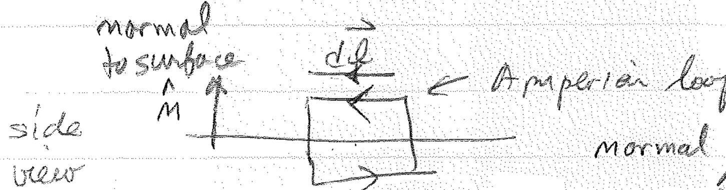
$$\oint \vec{B} \cdot d\vec{a} = (B_{above} - B_{below}) \cdot \hat{n}\, da = 0$$

$$(\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{n} = 0$$

normal component $\vec{B}$ continuous

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$





normal to loop is $\hat{n} \times d\vec{l}$ (out of page)

choose $d\vec{l}$ tangent to surface

$$(\vec{B}_{above} \cdot d\vec{l} - \vec{B}_{below} \cdot d\vec{l}) = \mu_0 \vec{K} \cdot (\hat{n} \times d\vec{l})$$

tangential component $\perp$ to $\vec{K}$ has jump

$$(\vec{B}_{above} - \vec{B}_{below}) \cdot d\vec{l} = \mu_0 (\vec{K} \times \hat{n}) \cdot d\vec{l}$$

$$\Rightarrow \boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})}$$

contains both results

$$\hat{n} \cdot (\vec{B}_{above} - \vec{B}_{below}) = \mu_0 \hat{n} \cdot (K \times \hat{n}) = 0$$

as $\vec{K} \times \hat{n} \perp \hat{n}$

$$d\vec{l} \cdot (\vec{B}_{above} - \vec{B}_{below}) = \mu_0 \, d\vec{l} \cdot (\vec{K} \times \hat{n})$$

In terms of potential: $\boxed{\dfrac{\partial \vec{A}_{above}}{\partial n} - \dfrac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}}$