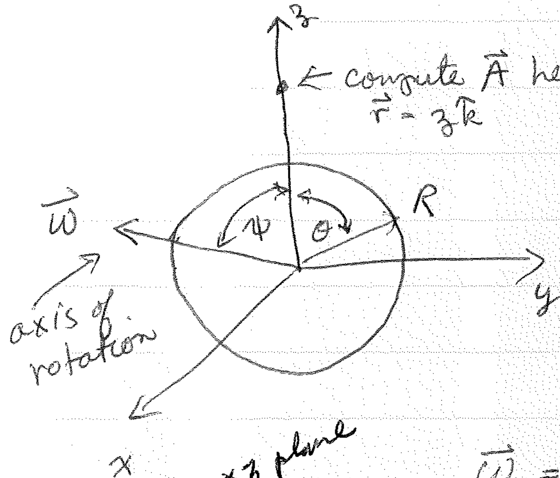
$\vec{B}$ Field of uniformly rotating charged sphere of radius R
surface charge σ



$\vec{\omega}$ is in the xz plane

$\Rightarrow$ surface current

$$\underbrace{\vec{K}(\theta,\phi)}_{\text{coords on surface of sphere}} = \sigma \underbrace{\vec{v}(\theta,\phi)}_{\text{velocity of surface at coords } (\theta,\phi)} = \sigma\, \vec{\omega} \times \underbrace{\vec{r}\,'}_{\text{pt on surface at } (\theta,\phi)}$$

$$\vec{\omega} = \omega(\cos\psi\, \hat{z} + \sin\psi\, \hat{x})$$

$$\vec{r}\,' = R(\cos\theta\, \hat{z} + \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y})$$

$$\vec{K} = \sigma\, \vec{\omega}\times\vec{r}\,' = \sigma\omega R\Big((\cos\psi\sin\theta\cos\phi - \sin\psi\cos\theta)\,\hat{y} - \cos\psi\sin\theta\sin\phi\, \hat{x} + \sin\psi\sin\theta\sin\phi\, \hat{z}\Big)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int da'\, \frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

here $\vec{r} = z\hat{z}$, $\quad \vec{r}\,' = R(\cos\theta\, \hat{z} + \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y})$

$$|\vec{r}-\vec{r}\,'| = \left(r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'\right)^{1/2}$$

$$= \left(z^2 + R^2 - 2zR\cos\theta\right)^{1/2}$$

$$da' = R^2\sin\theta\, d\theta\, d\phi$$

vanish when do $\int_0^{2\pi} d\phi$

$$A(z\hat{z}) = \frac{\mu_0\sigma\omega R^3}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, \frac{\left[(\cos\psi\sin\theta\cos\phi - \sin\psi\cos\theta)\,\hat{y} - \cos\psi\sin\theta\sin\phi\, \hat{x} + \sin\psi\sin\theta\sin\phi\, \hat{z}\right]}{(z^2 + R^2 - 2zR\cos\theta)^{1/2}}$$

$$\vec{A}(z\hat{k}) = \frac{\mu_0}{4\pi}\sigma\omega R^3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{(-\sin\psi\cos\theta\,\hat{y})}{(z^2+R^2-2zR\cos\theta)^{1/2}}$$

let $\mu = -\cos\theta$, $d\mu = \sin\theta\, d\theta$ $\qquad \int d\phi = 2\pi$

$$= +\frac{\mu_0}{2}\sigma\omega R^3 \sin\psi\,\hat{y} \underbrace{\int_{-1}^{1} d\mu \frac{\mu}{(z^2+R^2+2zR\mu)^{1/2}}}$$

integrate by parts

$$= \left[\mu \frac{(z^2+R^2+2zR\mu)^{1/2}}{zR}\right]_{-1}^{1} - \int_{-1}^{1} d\mu \frac{(z^2+R^2+2zR\mu)^{1/2}}{zR}$$

$$= \left[\mu \frac{(z^2+R^2+2zR\mu)^{1/2}}{zR}\right]_{-1}^{1} - \left[\frac{(z^2+R^2+2zR)^{3/2}}{3z^2R^2}\right]_{-1}^{1}$$

$$= \frac{(z^2+R^2+2zR)^{1/2} + (z^2+R^2-2zR)^{1/2}}{zR} - \frac{(z^2+R^2+2zR)^{3/2} - (z^2+R^2-2zR)^{3/2}}{3z^2R^2}$$

for $z > R$ outside:

$$= \frac{z+R+(z-R)}{zR} - \frac{(z+R)^3-(z-R)^3}{3z^2R^2}$$

$$= \frac{2z}{zR} - \left(\frac{z^3+R^3+3z^2R+3zR - z^3+R^3+3z^2R-3zR^2}{3z^2R^2}\right)$$

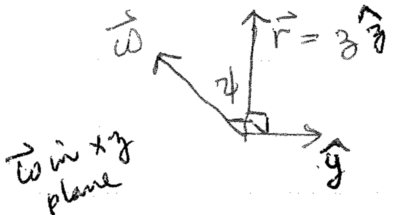
$$= \frac{2z}{zR} - \frac{2R^3+6z^2R}{3z^2R^2} = \frac{2}{R} - \frac{2}{3}\frac{R}{z^2} - \frac{2}{R} = -\frac{2}{3}\frac{R}{z^2}$$

for $z < R$ inside

$$= \frac{2R}{zR} - \frac{2z^3+6R^2z}{3z^2R^2} = \frac{2}{z} - \frac{2}{3}\frac{z}{R^2} - \frac{2}{z} = -\frac{2}{3}\frac{z}{R^2}$$

$$\vec{A}(z\hat{k}) = -\frac{\mu_0}{3}\sigma\omega R^3 \sin\psi\, \hat{j} \times \begin{cases} R/z^2 & \text{outside} \\ z/R^2 & \text{inside} \end{cases}$$

$$= \begin{cases} -\dfrac{\mu_0}{3}\dfrac{\sigma\omega R^4 \sin\psi}{z^2}\,\hat{y} & \text{outside} \\[2ex] -\dfrac{\mu_0}{3}\sigma\omega z R \sin\psi\,\hat{y} & \text{inside} \end{cases}$$

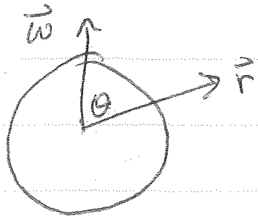


$$-\omega z \sin\psi\,\hat{y} = \vec{\omega}\times\vec{r}$$

$\Rightarrow$ without reference to any particular coordinate system

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0}{3}\dfrac{\sigma R^4}{r^3}\,\vec{\omega}\times\vec{r} & \text{outside} \\[2ex] \dfrac{\mu_0}{3}\sigma R\,\vec{\omega}\times\vec{r} & \text{inside} \end{cases}$$

Now compute $\vec{B} = \vec{\nabla}\times\vec{A}$.
Easiest to do this in spherical coords, where we now choose $\vec{\omega}$ along z axis,



$$\vec{\omega}\times\vec{r} = \omega r \sin\theta\,\hat{\phi}$$

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0}{3}\dfrac{\sigma R^4}{r^2}\omega\sin\theta\,\hat{\phi} & \text{outside} \\[2ex] \dfrac{\mu_0}{3}\sigma R r\omega\sin\theta\,\hat{\phi} & \text{inside} \end{cases}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

here $A_r = A_\theta = 0$, $A_\phi \neq 0$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)\hat{\theta}$$

<u>outside</u>

$$= \frac{\mu_0 \sigma R^4 \omega}{3}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\sin\theta}{r^2}\right)\hat{\theta}\right\}$$

$$\left\{\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right\}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 \sigma R^4 \omega}{3} \times \left\{\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right\}$$

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 \sigma R^4 \omega}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

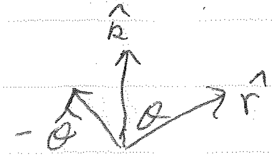
same functional form as for electric dipole!

<u>inside</u>

$$= \frac{\mu_0 \sigma R\,\omega}{3}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\, r\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\sin\theta\right)\hat{\theta}\right\}$$

$$\left\{\frac{2\sin\theta\cos\theta}{\sin\theta}\hat{r} - 2\sin\theta\,\hat{\theta}\right\}$$

$$= \frac{2}{3}\mu_0\sigma R\omega\left\{\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{\hat{z}}\right\}$$



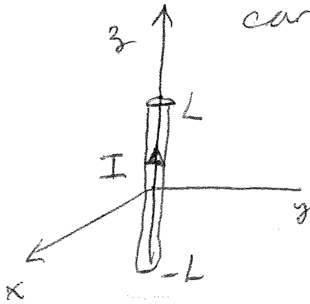
since $\vec{\omega} = \omega\hat{z}$

$$\boxed{\vec{B}(\vec{r}) = \frac{2}{3}\mu_0\sigma R\,\vec{\omega}}$$

$\vec{B}$ is uniform inside sphere and alligned with axis of rotation.

5.25

Find $\vec{A}$ of finite segment of straight wire carrying current I



problem is artificial since current not conserved at ends of wire. But we cant do infinite wire case since

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

only works when $\vec{J}(\vec{r}') \to 0$ as $\vec{r}' \to \infty$ – this isn't true for infinite wire. Nevertheless, we can do finite wire problem and see if we get the correct answer for $\vec{B}$ when we let $L \to \infty$ at end of calculation.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int dl' \, \frac{\vec{I}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

here, $\vec{I}(\vec{r}') = I\hat{z}$, $dl' = dz$ from $-L$ to $+L$

$\vec{r}' = z\hat{z}$

$\vec{r} = r\hat{r}$ in spherical coords

$\Rightarrow |\vec{r}-\vec{r}'| = \left(r^2+z^2-2rz\cos\theta\right)^{1/2}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{-L}^{L} dz \, \frac{I\hat{z}}{\left(r^2+z^2-2rz\cos\theta\right)^{1/2}} =$$

$$= \frac{\mu_0}{4\pi} I\hat{z} \, \ln\left[2\sqrt{r^2+z^2-2rz\cos\theta} + 2z - 2r\cos\theta\right]_{-L}^{L}$$

$$A(\vec{r}) = \frac{\mu_0}{4\pi} I \hat{z} \Big\{ \ln\left[2\sqrt{L^2+r^2-2rL\cos\theta} + 2L - 2r\cos\theta\right]$$
$$- \ln\left[2\sqrt{L^2+r^2+2rL\cos\theta} - 2L - 2r\cos\theta\right]$$

consider $L \gg r$ so that observer won't see unphysical ends of the wire.

$$2\sqrt{L^2+r^2-2rL\cos\theta} + 2L - 2r\cos\theta$$
$$= 2L\sqrt{1+\left(\frac{r}{L}\right)^2 - 2\frac{r}{L}\cos\theta} + 2L - 2r\cos\theta \qquad \sqrt{1+\epsilon} \cong 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}$$

$$= 2L\left(1 - \frac{r}{L}\cos\theta + \frac{1}{2}\left(\frac{r}{L}\right)^2 - \frac{1}{2}\left(\frac{r}{L}\right)^2\cos^2\theta + O\left(\frac{r}{L}\right)^3\right) + 2L - 2r\cos\theta$$

$$= 2L - 2r\cos\theta + \frac{r^2}{L} - \frac{r^2}{L}\cos^2\theta + 2L - 2r\cos\theta$$

$$= 4L - 4r\cos\theta + \frac{r^2}{L}(1-\cos^2\theta) = 4L - 4r\cos\theta + \frac{r^2}{L}\sin^2\theta$$

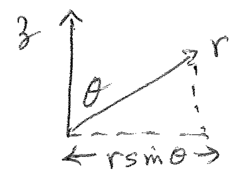
similarly: $2\sqrt{L^2+r^2+2rL\cos\theta} - 2L - 2r\cos\theta$

$$= 2L\left(1 + \frac{r}{L}\cos\theta + \frac{1}{2}\left(\frac{r}{L}\right)^2 - \frac{1}{2}\left(\frac{r}{L}\right)^2\cos^2\theta + O\left(\frac{r}{L}\right)^3\right) - 2L - 2r\cos\theta$$

$$= 2L + 2r\cos\theta + \frac{r^2}{L} - \frac{r^2}{L}\cos^2\theta - 2L - 2r\cos\theta$$
$$= \frac{r^2}{L}(1-\cos^2\theta) = \frac{r^2}{L}\sin^2\theta$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \hat{z} \ln\left[\frac{4L - 4r\cos\theta + \frac{r^2}{L}\sin^2\theta}{\frac{r^2}{L}\sin^2\theta}\right]$$

$$= \frac{\mu_0}{4\pi} I \hat{z} \ln\left[\frac{4 - 4\left(\frac{r}{L}\right)\cos\theta + \left(\frac{r}{L}\right)^2\sin^2\theta}{\left(\frac{r}{L}\right)^2\sin^2\theta}\right]$$



As $L \to \infty$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \hat{z} \ln\left(\frac{4L^2}{r^2 \sin^2\theta}\right)$$

note $r\sin\theta$ = projection of $\vec{r}$ into xy plane = cylindrical coord "r"

$$\underbrace{\vec{A}(r,\phi,z)}_{\text{cylindrical coords}} = \frac{\mu_0}{2\pi} I \hat{z} \ln\left(\frac{2L}{r}\right)$$

evaluate $\vec{\nabla} \times \vec{A}$ in cylindrical coords, where $A_r = A_\phi = 0$
A_z depends only on r

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial r} \hat{\phi}$$

$$= -\frac{\mu_0}{2\pi} I \underbrace{\frac{\partial}{\partial r}\left(\ln\frac{2L}{r}\right)}_{-\frac{1}{r}} \hat{\phi}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}}$$

agree's with earlier result!

Multipole expansion for $\vec{A}(\vec{r})$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

we saw when doing the multipole expansion for $V(\vec{r})$ that

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

So

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi r} \sum_{n=0}^{\infty} \int d^3r' \, \vec{j}(\vec{r}') \left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$n=0$ monopole term

$$\vec{A}_{mono}(\vec{r}) = \frac{\mu_0}{4\pi r} \int d^3r' \, \vec{j}(\vec{r}')$$

use $\vec{\nabla}\cdot(r_\mu \vec{j}) = r_\mu \vec{\nabla}\cdot\vec{j} + \vec{j}\cdot\vec{\nabla} r_\mu$ — $\vec{\nabla} r_\mu = \hat{\mu}$ unit vector in direction μ

$= r_\mu \vec{\nabla}\cdot\vec{j} + j_\mu$ — example $\frac{\partial}{\partial x}(x,y,z) = (1,0,0) = \hat{x}$

$= -r_\mu \frac{\partial \rho}{\partial t} + j_\mu$ — using $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$

So

$$\int d^3r \, j_\mu(\vec{r}) = \int d^3r \left[\vec{\nabla}\cdot(r_\mu \vec{j}) + r_\mu \frac{\partial \rho}{\partial t}\right]$$

$$= \underbrace{\oint_S d\vec{a}\cdot r_\mu \vec{j}}_{\to 0} + \int d^3r \, r_\mu \frac{\partial \rho}{\partial t}$$

assuming $S \to \infty$ and current $\vec{j}$ is localized

$$= \frac{d}{dt} \int d^3r \, r_\mu \rho = \frac{dp_\mu}{dt}$$

p_μ is μ component of electric dipole moment

$= 0$ since in statics $\frac{d\vec{p}}{dt} = 0$

$\Rightarrow \int d^3r\, \vec{J}(\vec{r}) = 0$ in magnetostatics

monopole term always vanishes!

<u>n=1 dipole term</u>

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\int d^3r'\, \vec{J}(\vec{r}')\, r'\cos\theta = \frac{\mu_0}{4\pi r^2}\int d^3r'\, \vec{J}(\vec{r}')\,(\vec{r}'\cdot\hat{r})$$

$$= \frac{\mu_0}{4\pi r^2}\,\hat{r}\cdot\int d^3r' \underbrace{\left[\vec{r}'\,\vec{J}(\vec{r}')\right]}_{\text{tensor}}$$

we need to compute the tensor $\int d^3r\; r_\mu j_\nu \qquad \mu,\nu = x,y,z$

use $\vec{\nabla}\cdot(r_\mu r_\nu \vec{J}) = r_\mu \vec{\nabla}\cdot(r_\nu\vec{J}) + (r_\nu\vec{J})\cdot\vec{\nabla}r_\mu = r_\mu j_\nu + r_\nu j_\mu$

$$\int d^3r\; r_\mu j_\nu = \int d^3r\left[\vec{\nabla}\cdot(r_\mu r_\nu\vec{J}) - r_\nu j_\mu\right]$$

(from monopole calc $\vec{\nabla}\cdot(r_\nu\vec{J}) = j_\nu$ in magnetostatics)

$$= \underbrace{\oint_S d\vec{a}\cdot(r_\mu r_\nu\vec{J})}_{\to 0 \text{ as } S\to\infty \text{ for localized } \vec{J}} - \int d^3r\; r_\nu j_\mu$$

$\Rightarrow \int d^3r\; r_\mu j_\nu = -\int d^3r\; r_\nu j_\mu \qquad$ tensor is antisymmetric

$$\int d^3r\; r_\mu j_\nu = \frac{1}{2}\int d^3r\left[r_\mu j_\nu - r_\nu j_\mu\right]$$

or in tensor form $\int d^3r\; \vec{r}\,\vec{J} = \frac{1}{2}\int d^3r\left[\vec{r}\,\vec{J} - \vec{J}\,\vec{r}\right]$

where $[\vec{r}\,\vec{J}]_{\mu\nu} = r_\mu j_\nu$

Now $\hat{r}\cdot\int d^3r'\left[\vec{r}'\vec{J}(\vec{r}')\right] = \frac{1}{2}\int d^3r'\ \hat{r}\cdot\left[\vec{r}'\vec{J} - \vec{J}\vec{r}'\right]$

$$= \frac{1}{2}\int d^3r'\left[(\hat{r}\cdot\vec{r}')\vec{J} - (\hat{r}\cdot\vec{J})\vec{r}'\right]$$

triple product rule
$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$= \frac{1}{2}\int d^3r'\ \hat{r}\times(\vec{J}\times\vec{r}')$$

So $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\ \hat{r}\times\frac{1}{2}\int d^3r'\ \vec{J}\times\vec{r}' = \frac{-\mu_0}{4\pi r^2}\ \hat{r}\times\frac{1}{2}\int d^3r'\ \vec{r}'\times\vec{J}$

define the magnetic dipole moment

$$\boxed{\vec{m} \equiv \frac{1}{2}\int d^3r'\left[\vec{r}'\times\vec{J}(\vec{r}')\right]}$$

$$\vec{A}_{dip}(\vec{r}) = \frac{-\mu_0}{4\pi r^2}\ \hat{r}\times\vec{m} = \boxed{\frac{\mu_0}{4\pi r^2}\ \vec{m}\times\hat{r} = \vec{A}_{dip}(\vec{r})}$$

For a wire loop $\vec{m} = \frac{1}{2} I \oint \vec{r}'\times d\vec{l}$ using $d\vec{I} = I d\vec{l}$

$\vec{m}$ is independent of the choice of the origin

If we shift origin by transforming to new coordinate system
$\tilde{\vec{r}} = \vec{r} + \vec{d}$
then in the $\tilde{\vec{r}}$ coordinate system

$$\tilde{\vec{m}} = \frac{1}{2}\int d^3\tilde{r}\,(\tilde{\vec{r}}\times\vec{J}) = \frac{1}{2}\int d^3r\ (\vec{r}+\vec{d})\times\vec{J}$$

$$= \frac{1}{2}\int d^3r(\vec{r}\times\vec{J}) + \frac{1}{2}\vec{d}\times\underbrace{\int d^3r\ \vec{J}}_{=0}$$

we have seen that 2nd term vanishes in magnetostatics

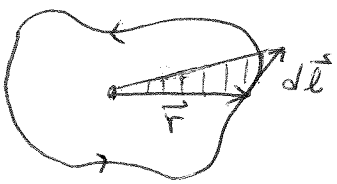
$$\tilde{\vec{m}} = \vec{m}$$

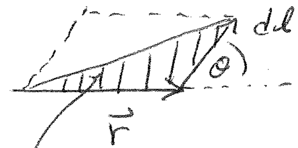
For a <u>planar loop</u>:

$$\vec{m} = \frac{1}{2} I \oint \vec{r} \times d\vec{l}$$

$\vec{r} \times d\vec{l} = r\, dl \sin\theta\, \hat{n}$

↑ normal to plane — use right hand rule for direction



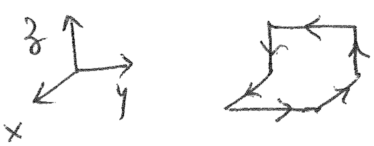


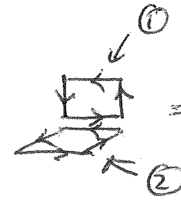
area is $(r\, dl \sin\theta)(\frac{1}{2})$

$$\Rightarrow \vec{m} = \hat{n} I \oint dl \frac{r \sin\theta}{2} = \hat{n}\, I\, (\text{area of loop})$$

if $\vec{a} \equiv \text{area}\ \hat{n}$ then $\vec{m} = I\vec{a}$

Can use this to get $\vec{m}$ for a <u>piecewise planar loop</u>
for example:



= superposition of  $\Rightarrow \vec{m}_1 + \vec{m}_2 = \vec{m}$

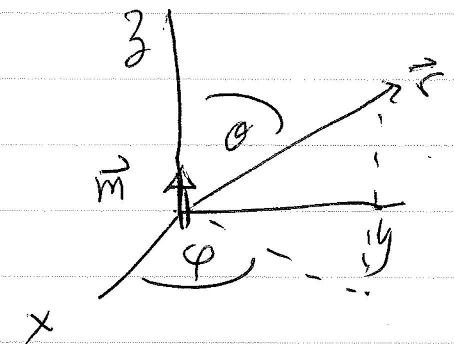
where $\vec{m}_1 = I a_1 \hat{x}$, $I a_2 \hat{z} = \vec{m}_2$

where a_1 and a_2 are areas of the two pieces.

Magnetic field in dipole approximation

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \nabla \times \left[\frac{\mu_0}{4\pi r^2} (\vec{m} \times \hat{r}) \right]$$

write in spherical coordinates choose $\hat{z}$ axis to lie along $\vec{m}$



$$\vec{m} \times \hat{r} = m \sin\theta \, \hat{\varphi}$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi r^2} m \sin\theta \, \hat{\varphi}$$

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} (\sin\theta \, A_\varphi) \right] \hat{r} - \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\varphi) \right] \hat{\theta}$$

since $A_r = A_\theta = 0$

$$\vec{B}_{dip}(\vec{r}) = \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin^2\theta \frac{\mu_0 m}{4\pi r^2} \right) \right] \hat{r} - \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\mu_0}{4\pi r^2} m \sin\theta \right) \right] \hat{\theta}$$

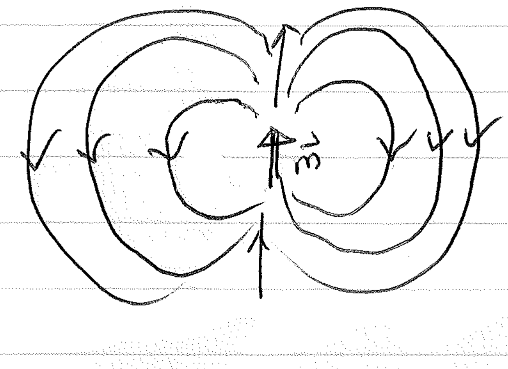
$$= \frac{1}{r \sin\theta} 2 \sin\theta \cos\theta \frac{\mu_0 m}{4\pi r^2} \hat{r} + \frac{1}{r^3} \frac{\mu_0}{4\pi} m \sin\theta \, \hat{\theta}$$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3} \left[2 \cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right]$$

← same form as $\vec{E}$ field from electric dipole $\vec{p}$

$$= \frac{\mu_0}{4\pi r^3} \left[3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} \right]$$

← just like we found for $\vec{E}_{dip}$, expresses $\vec{B}_{dip}$ without reference to any particular coordinate system



$\vec{B}$ field lines for dipole $\vec{m}$

5.37

Circular loop of radius R in xy plane, current I.

$$\Rightarrow \quad \vec{m} = I \text{ area } \hat{z} = I \pi R^2 \hat{z}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{I \pi R^2}{r^3} \left(2 \cos\theta \, \hat{r} + \sin\theta \, \hat{\theta} \right)$$

along z axis, $\theta = 0$ for $z > 0$
$\theta = \pi$ for $z < 0$

$$\vec{B}(z\hat{z}) = \frac{\mu_0 I R^2}{4 z^3} 2 \hat{z} \quad \text{for } z > 0$$

$$\frac{\mu_0 I R^2}{4 z^3} (-2)(-\hat{z}) \quad \text{for } z < 0$$

$$\vec{B}(z\hat{z}) = \frac{\mu_0 I R^2}{2 z^3} \hat{z} \quad \text{for both } z > 0 \text{ and } z < 0$$

exact solution (Ex 6) was

$$\vec{B}(z\hat{k}) = \frac{\mu_0 I}{2} \frac{R^2 \hat{z}}{(R^2 + z^2)^{3/2}}$$

for $z \gg R$, $(R^2 + z^2)^{3/2} \sim z^3$, so exact answer agrees with dipole approx in far field limit $z \gg R$