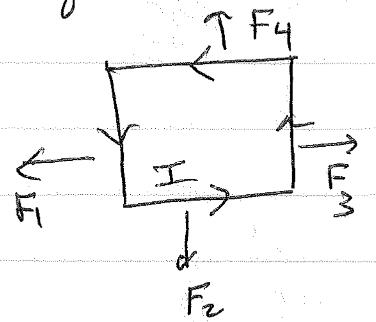
Force and torque on a magnetic dipole in an external magnetic field.

Consider as a model magnetic dipole a square loop of side a with current I circulating



⊙ $\vec{B}$ out of page is uniform

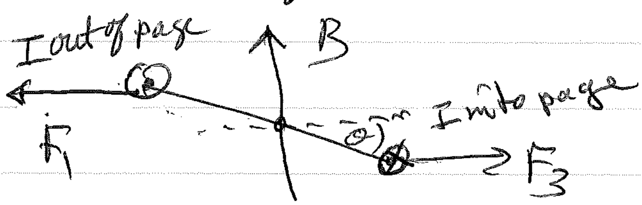
Force is $a\vec{I}\times\vec{B}$ on each side of loop

The magnitude of the force F_1, F_2, F_3, F_4 on each side will be equal, and so they cancel out and net force is zero.

But if $\vec{B}$ is non-uniform, the force on one side will be greater than the force on the opposing side, and there will be a net force. $\vec{F}=0$ if $\vec{B}$ uniform, $\vec{\nabla}\vec{B}=0$

$\vec{F}\neq 0$ if $\vec{B}$ varies, $\vec{\nabla}\vec{B}\neq 0$

If the loop is oriented at an angle in a uniform $\vec{B}$, then the net force is zero, but there will be a net torque



side view of loop

$\vec{F}_2$ and $\vec{F}_4$ still cancel and give no torque.

$\vec{F}_1$ and $\vec{F}_3$ cancel but give net torque acting to align the loop so that the dipole moment $\vec{m}$ is $\parallel$ to $\vec{B}$.

Force

We can get expression for net force $\vec{F}$ and torque $\vec{N}$ for a generalized localized current distribution $\vec{j}(\vec{r})$.

$$\vec{F} = \int d^3r \; \vec{j}(\vec{r}) \times \vec{B}(\vec{r})$$

localized current $\vec{j}$ in external $\vec{B}$

assume that $\vec{B}$ is slowly varying in space over the length that $\vec{j}$ is localized. Choose the origin of the coordinates to be in the middle of $\vec{j}$ and then expand $\vec{B}(\vec{r})$ about the origin

$$\vec{B}(\vec{r}) \simeq \vec{B}(0) + (\vec{r}\cdot\vec{\nabla})\vec{B}(0) + \cdots$$

$$\vec{F} = \Big[\int d^3r \; \vec{j}(\vec{r})\Big] \times \vec{B}(0) + \int d^3r \; \vec{j}(\vec{r}) \times (\vec{r}\cdot\vec{\nabla})\vec{B}(0)$$

$\Big[\int d^3r \; \vec{j}(\vec{r})\Big] = 0$ in magnetostatics. So if $\vec{B}$ is uniform then $(\vec{r}\cdot\vec{\nabla})\vec{B} = 0$ and $\vec{F} = 0$. But if $\vec{B}$ is not uniform then

$$\vec{F} = \int d^3r \; \vec{j}(\vec{r}) \times (\vec{r}\cdot\vec{\nabla})\vec{B}(0)$$

Consider in terms of components

$$F_i = \varepsilon_{ikl} \int d^3r \; j_k \, r_m \frac{\partial}{\partial r_m} B_l$$

ε_{ikl} is Levi-Civita symbol
sum over k, l, m implied

recall from derivation of magnetic dipole approx we had

$$\int d^3r \; j_k r_m = -\int d^3r \; r_k j_m = \frac{1}{2}\int d^3r \left[j_k r_m - r_k j_m \right]$$

Now we can write

$$\frac{1}{2}\int d^3r \left[j_k r_m - r_k j_m \right] = -m_n \varepsilon_{nkm}$$

follows from $\vec{m} = \frac{1}{2}\int d^3r\ \vec{r}\times\vec{j}$

so $\quad m_n = \frac{1}{2}\varepsilon_{nil}\int d^3r\ r_i j_l$

so $\quad m_n \varepsilon_{nkm} = \frac{1}{2}\varepsilon_{nkm}\varepsilon_{nil}\int d^3r\ r_i j_l$

$$= \frac{1}{2}\left[\delta_{ki}\delta_{ml} - \delta_{kl}\delta_{mi}\right]\int d^3r\ r_i j_l$$

$$= \frac{1}{2}\int d^3r\left[r_k j_m - r_m j_k\right]$$

So

$$F_i = \varepsilon_{ikl}\varepsilon_{nkm}(-m_n)\frac{\partial B_l}{\partial r_m}$$

$$= -(\delta_{in}\delta_{lm} - \delta_{im}\delta_{nl})\, m_n \frac{\partial B_l}{\partial r_m}$$

$$= -m_i \underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0} + \frac{\partial}{\partial r_i}(\vec{m}\cdot\vec{B})$$

can bring m_n inside the derivative since it is constant

So $\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})}$

Torque

force per unit volume on $\vec{j}$

$$\vec{N} = \int d^3r \; \vec{r} \times (\vec{j} \times \vec{B})$$

to lowest order take $\vec{B}$ to be uniform

$$= \int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{B}\,(\vec{r}\cdot\vec{j}) \right]$$

by triple product rule
$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

2nd term $= 0$ as follows

$$\int d^3r \; (\vec{r}\cdot\vec{j}) = \int d^3r \; \vec{j}\cdot\vec{\nabla}\left(\frac{r^2}{2}\right) \qquad \text{since } \vec{\nabla}\left(\frac{r^2}{2}\right) = \vec{r}$$

$$= -\int d^3r \, (\vec{\nabla}\cdot\vec{j})\left(\frac{r^2}{2}\right)$$

integrating by parts
surface term $\to 0$ since
$\vec{j}$ is localized.

$$= 0 \qquad \text{since } \vec{\nabla}\cdot\vec{j} = 0 \text{ in magnetostatics}$$

1st term is

$$\int d^3r \, \vec{j}\,\vec{r} = -\int d^3r \, \vec{r}\,\vec{j} = \frac{1}{2}\int d^3r \left[\vec{j}\,\vec{r} - \vec{r}\,\vec{j} \right]$$

$$\vec{N} = \frac{1}{2}\left[\int d^3r \, \vec{j}\,\vec{r}\right]\cdot\vec{B} = \frac{1}{2}\int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B}) \right]$$

$$= \frac{1}{2}\int d^3r \, (\vec{r}\times\vec{j})\times\vec{B} \qquad \text{by triple product rule}$$

$$\boxed{N = \vec{m}\times\vec{B}}$$

Magnetism in Matter

Atoms in a material can develop magnetic moments when a $\vec{B}$ is turned on.
Such moments arise from orbital motion of electrons around nucleus, and from intrinsic magnetic moments due to the intrinsic spin of electrons and nuclei.
See text for classical discussion of such effects, although really need quantum mechanics to explain properly.

Materials fall into one of three categories

1) paramagnetic: net magnet moment vanishes when $\vec{B} = 0$ but when $\vec{B} \neq 0$, material develops a magnetic moment parallel to $\vec{B}$.

2) diamagnetic: net magnetic moment vanishes when $\vec{B} = 0$ but when $\vec{B} \neq 0$, material develops a magnetic moment antiparallel to $\vec{B}$

3) ferromagnetic: there can be a net magnetic moment in the material even when $\vec{B} = 0$

Magnetization density. When apply $\vec{B}$, develope local atomic magnetic moments

$\vec{M}$ = magnetic dipole moment per unit volume

$$\int_V d^3r\, \vec{M}(\vec{r}) = \text{total magnetic dipole moment in volume } V$$

Vector potential produced by magnetization $\vec{M}(r)$

in dipole approx: (see 5.79) $\qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$ dipole at origin

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{M}(\vec{r}\,') \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, d^3r'$$

$$= \frac{\mu_0}{4\pi} \int_V \vec{M}(r') \times \vec{\nabla}'\left(\frac{1}{|\vec{r} - \vec{r}\,'|}\right) d^3r'$$

integrate by parts

$\vec{\nabla} \times (f\vec{A}) = f\,\vec{\nabla} \times \vec{A} - \vec{A} \times \vec{\nabla} f$

$$= \frac{\mu_0}{4\pi}\left[\int_V \frac{1}{|\vec{r} - \vec{r}\,'|} \vec{\nabla}' \times \vec{M}(\vec{r}\,')\, d^3r' - \int_V \vec{\nabla}' \times \left(\frac{\vec{M}(r')}{|r - r'|}\right) d^3r'\right]$$

$$= \frac{\mu_0}{4\pi} \int \frac{1}{|\vec{r} - \vec{r}\,'|} \vec{\nabla}' \times \vec{M}(\vec{r}\,')\, d^3r' + \frac{\mu_0}{4\pi} \oint_S \frac{1}{|r - r'|} \vec{M}(r') \times d\vec{a}'$$

(see prob 1.61b)

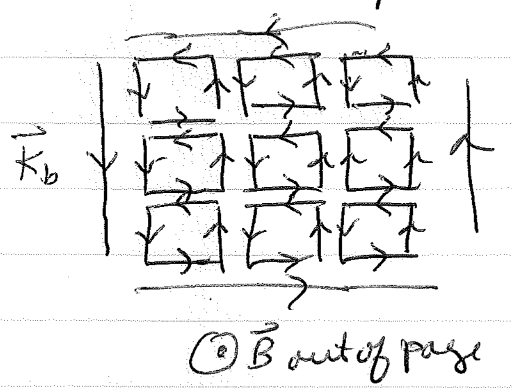
define $\quad \vec{J}_b = \vec{\nabla} \times \vec{M} \quad$ bound current density

$\vec{K}_b \equiv \vec{M} \times \hat{n} \quad$ bound surface current density

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3r' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b\, da'}{|\vec{r} - \vec{r}\,'|}$$

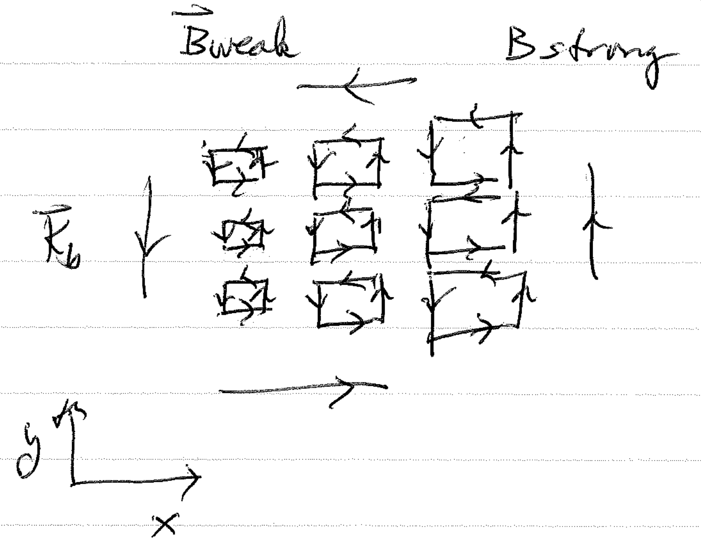
Simple picture for understanding $\vec{K}_b$ and $\vec{J}_b$

Uniform $\vec{B}$ induces uniform $\vec{M}$
model induced magnetic dipoles like small current loops



We see that inside the material, the currents from adjacent loops cancels, so $\vec{J}_b = 0$ inside. But a net surface current $\vec{K}_b$ remains. With $\vec{M} \parallel \vec{B}$ we see that the direction of $\vec{K}_b$ is indeed given by $\vec{M} \times \hat{n}$, with $\hat{n}$ the outward normal.

Non uniform $\vec{B}$ induces non-uniform $\vec{M}$



Now there is a surface current $\vec{K}_b$ as before, but also there is a $\vec{J}_b$ inside. Current on horizontal segments of adjacent loops will cancel, but current on vertical segments of adjacent loops will not cancel. There is net $\vec{J}_b$ in the $-\hat{y}$ direction

For $\vec{B} = \vec{B}(x)\hat{z}$ we expect $\vec{M} = M(x)\hat{z}$ and

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = -\frac{\partial M}{\partial x}\hat{y}$$

potential from $\vec{M}$ exact as if there were current sources $\vec{J}_b$ and $\vec{K}_b$.

Note $\vec{\nabla}\cdot\vec{J}_b = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) = 0$, as it must if $\vec{J}_b$ is to be a magnetostatic current

Above dipole approx good far away from magnetic dipoles. But we want to apply the result also inside the material, ie close to dipoles. This is same problem we had in treating electric potential V from polarization density $\vec{P}$. Similarly we find in magnetostatics, that if we ~~treat~~ consider the ~~average or~~ macroscopic $\vec{B}$ field averaged over length scales large on atomic, but small on system size, then the above dipole approx does correctly give this macroscopic B field even inside the material.

(see prob 6.11 and 5.48)

H- field

Amperes law: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{total} = \mu_0(\vec{J}_{free} + \vec{J}_b)$

bound currents due to $\vec{M}$

"free" currents added by experimentor

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \vec{\nabla}\times\vec{M}$$

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_{free} \quad \Rightarrow \quad \vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_{free}} \Rightarrow \oint \vec{H}\cdot d\vec{l} = I_{free}^{encl}$$

ex: ferromagnet

$\vec{\nabla}\cdot\vec{M} \neq 0$ on top and bottom surfaces

Note:

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = -\vec{\nabla}\cdot\vec{M} \neq 0 \text{ in general}$$

So we cannot simply replace $\vec{B}$ by $\vec{H}$ and $\vec{J}$ by $\vec{J}_{free}$ and reduce magnetostatics in matter to magnetostatics in free space. Only can do this if problem is such that $\vec{\nabla}\cdot\vec{M} = 0$, as might be the case from some symmetry.

Maxwell's eqn for magnetostatics in matter

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

compare to:

$$\left(\begin{array}{l}\vec{\nabla}\cdot\vec{D} = \rho_{free} \\ \vec{\nabla}\times\vec{E} = 0\end{array}\right)$$

<u>Linear magnetic media</u> (good for para or dia magnet; not true for ferromagnet)

$$\boxed{\vec{M} = \chi_m \vec{H}}$$

χ_m is magnetic susceptibility $\begin{cases}\chi_m > 0 & \text{param} \\ \chi_m < 0 & \text{diam}\end{cases}$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(\vec{H} + \chi_m\vec{H}) = \mu_0(1+\chi_m)\vec{H}$$

$$\boxed{\vec{B} = \mu\vec{H}}$$

where $\mu = \mu_0(1+\chi_m)$ is <u>permeability</u> of the material

$\vec{M} = \left(\frac{\chi_m}{\mu}\right)\vec{B}$

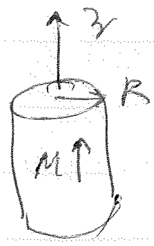
for linear material, $\vec{\nabla}\cdot\vec{M} = \vec{\nabla}\cdot(\chi_m\vec{H}) = \vec{\nabla}\cdot\left(\frac{\chi_m}{\mu}\vec{B}\right)$

$= \frac{\chi_m}{\mu}\vec{\nabla}\cdot\vec{B} = 0$ provided one stays <u>inside</u> material where $\frac{\chi_m}{\mu}$ = constant

but $\vec{\nabla}\cdot\vec{M} \neq 0$ at boundary between two different media, in general bound current

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = (\vec{\nabla} \times \chi_m \vec{H}) = \chi_m \vec{\nabla} \times \vec{H} = \chi_m \vec{J}_{free}$$

$\Rightarrow$ unless free current flows through material, all bound currents are at surface.



Examples
prob 6.7

infinitely long cylinder

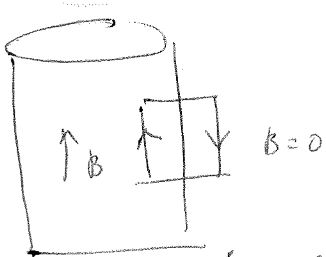
constant magnetization
$\vec{M} = M\hat{z}$

What is $\vec{B}$?

$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$ as $\vec{M}$ constant

Bound currents : $\vec{K}_b = \vec{M} \times \hat{n}$ $\quad \hat{n}$ = unit normal to surface $= \hat{r}$

$\vec{K}_b = M\hat{z} \times \hat{r} = M\hat{\phi}$ solenoidal surface current



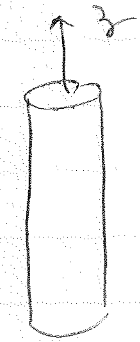
$\Rightarrow \vec{B} = \begin{cases} 0 & \text{outside} \\ \mu_0 M\hat{z} & \text{inside} \end{cases}$

$\oint B \cdot dl = Bl = \mu_0 l K_b$

inside $\vec{B} = \mu_0 K_b \hat{z}$

here $K_b = M$

Prob 6.12



Infinitely long cylinder

$\vec{M} = kr\hat{z}$ frozen in magnetization

bound currents:

$$\vec{J}_b = \nabla \times \vec{M} = -\frac{\partial M_z}{\partial r}\hat{\phi} \text{ in cylindrical coords}$$
$$= -k\hat{\phi}$$
$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{r} = kR\hat{\phi}$$

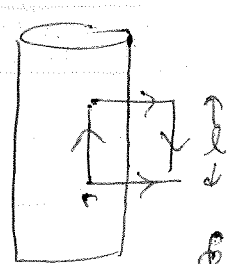
a) find $\vec{B}$ inside and outside, using the bound currents

Ampere: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$

Since all currents are solenoidal, we expect $\vec{B} = B(r)\hat{z}$

Also, expect $\vec{B} = 0$, $r > R$.

(View as superposition of many solenoids of radii $0 < r \leq R$)



$$\oint \vec{B}\cdot d\vec{l} = B(r)l = \mu_0 I_{enc}$$
$$= \mu_0 l \int_r^R dr' \underbrace{(-k)}_{\vec{J}_b(r)\cdot\hat{\phi}} + \mu_0 l \underbrace{kR}_{\vec{K}_b\cdot\hat{\phi}}$$

$$= \mu_0 l \{-k(R-r) + kR\} = \mu_0 l k r$$

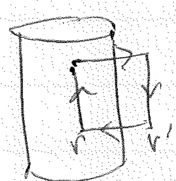
$$B(r) = \mu_0 k r$$

$$\text{So } \vec{B}(\vec{r}) = \begin{cases} \mu_0 k r \hat{z} & r \leq R \\ 0 & r > 0 \end{cases}$$

b) find $\vec{B}$ by solving first for $\vec{H}$

$$\oint \vec{H}\cdot d\vec{l} = I_{free}^{encl} = 0$$

since currents are all solenoidal, $\vec{H}$ is along $\hat{z}$ $\quad \vec{H} = H(r)\,\hat{z}$



$$\oint \vec{H}\cdot d\vec{l} = \left(H(r) - H(r')\right) l = 0$$

$\Rightarrow \vec{H}$ is constant

far from cylinder we expect $\vec{H}=0$

$\Rightarrow \vec{H}=0$ everywhere!

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \Rightarrow \quad \vec{B} = \mu_0\left(\vec{H}+\vec{M}\right)$$

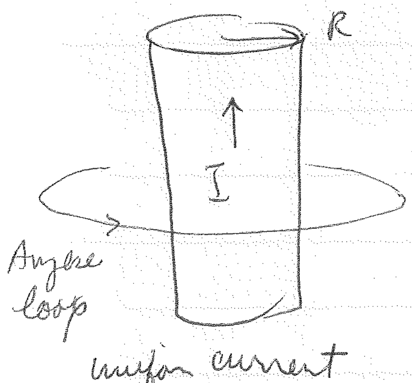
$$= \mu_0 \vec{M} \quad \text{as } \vec{H}=0$$

$$\vec{B} = \begin{cases} 0 & r>R \text{ outside} \\ \mu_0 k r \hat{z} & r<R \text{ inside} \end{cases}$$

same as from part (a)!
only much simpler!

prob 6.17

long straight wire, carries current I, distributed uniformly over cross sectional area of wire. Wire has χ_m. What is $\vec{B}(r)$? What are bound currents?



by symmetry, expect H is in $\hat{\phi}$ direction

$$\vec{H} = H(r)\hat{\phi}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = I_{en}^{free}$$ take loop at radius r

$$2\pi r\, H(r) = I_{en}^{free} = \begin{cases} I & r > R \\ I\left(\frac{r^2}{R^2}\right) & r < R \end{cases}$$ as I uniform

fraction of crosssectional area enclosed by loop

$$\vec{H}(\vec{r}) = \begin{cases} \frac{I}{2\pi r}\hat{\phi} & r > R \\ \frac{I r}{2\pi R^2}\hat{\phi} & r < R \end{cases}$$

$$\vec{B} = \mu \vec{H} \qquad \text{where } \mu = \mu_0(1+\chi_m)$$

$$\Rightarrow \vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 I}{2\pi r}\hat{\phi} & r > R \quad (\chi_m = 0 \text{ outside}) \\ \frac{\mu I r}{2\pi R^2}\hat{\phi} & r < R \quad (\chi_m \neq 0 \text{ inside}) \end{cases}$$

Magnetization $\vec{M} = \chi_m \vec{H} = \begin{cases} \frac{\chi_m I r}{2\pi R^2}\hat{\phi} & r < R \\ 0 & r > R \end{cases}$

bound current density $\vec{J}_b = \vec{\nabla} \times \vec{M}$

evaluate in cylindrical coords. Since $\vec{M}$ depends only on r + points only in $\hat{\phi}$ direction

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{r}\frac{\partial}{\partial r}(rM)\,\hat{z}$$

$$= \frac{1}{r}\,\frac{\chi_m I}{2\pi R^2}\,\frac{\partial}{\partial r}\left(r^2\right)\hat{z} = \frac{\chi_m I}{\pi R^2}\,\hat{z}$$

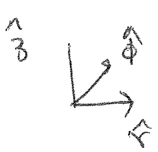
compare with $\vec{J}_{free} = \dfrac{I}{\pi R^2}\hat{z} \Rightarrow \vec{J}_b = \chi_m \vec{J}_f$ as expected

surface bound current

$$\vec{K}_b = \vec{M} \times \hat{n} \quad . \text{ Here } \hat{n} = \hat{r} \text{ cylindrical radial coord}$$

$$= \frac{\chi_m I R}{2\pi R^2}\,\hat{\phi} \times \hat{r}$$

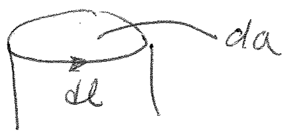
$$\hat{\phi} \times \hat{r} = -\hat{z}$$



$$= \frac{\chi_m I}{2\pi R}\left(-\hat{z}\right)$$

Net bound current flowing down wire is

$$\int \vec{J}_b \cdot d\vec{a} + \oint \vec{K}_b \cdot \hat{z}\, dl$$



$$= \pi R^2 \vec{J}_b \cdot \hat{z} + 2\pi R \vec{K}_b \cdot \hat{z}$$

since J_b is constant

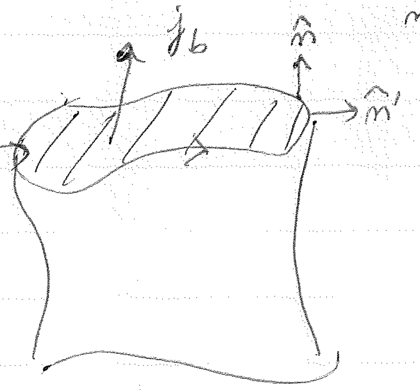
$$= \chi_m I + 2\pi R\,\frac{\chi_m I}{2\pi R}(-1)$$

$$= \chi_m I - \chi_m I = 0$$

no net bound current flows

True in general

flat crossectional area $\int da$ bounded by loop $\int dl$



$\hat{n} \perp$ to plane of area
$\hat{n}'$ normal to loop bounding area

$\hat{n}$ is direction down length of wire
$\hat{n}'$ is outward normal to wire

$$\int d\vec{a} \cdot \vec{j}_b + \int dl \; \vec{K}_b \cdot \hat{n}$$

$$= \int d\vec{a} \cdot (\vec{\nabla} \times \vec{M}) + \int dl \, (\vec{M} \times \hat{n}') \cdot \hat{n}$$

$$= \int d\vec{a} \cdot (\vec{\nabla} \times \vec{M}) + \int dl \, \underbrace{(\hat{n}' \times \hat{n})}_{-\hat{t} \text{ unit tangent}} \cdot \vec{M}$$

Stokes ↓ , $d\vec{l} = dl\,\hat{t}$

$$= \oint d\vec{l} \cdot \vec{M} - \int d\vec{l} \cdot \vec{M} = 0$$

net bound current flowing down wire is always zero
surface current always cancells bulk current

This is because bound currents come from local circulating atomic current loops – these local atomic loops cannot lead to any net charge traveling down the wire, ie the net total bound current must sum to zero.

Equivalent to result in dielectrics that net bound charge must sum to zero since the dielectric is neutral.