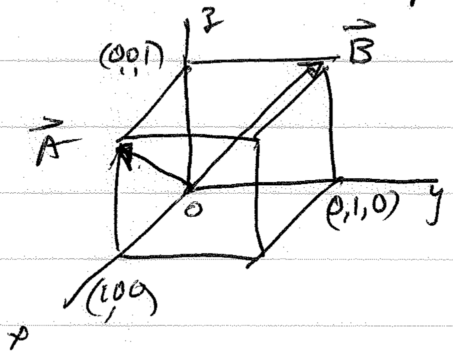
Workshop Week 1

1) Griffiths 1.2 Example – angle between face diagonals of cube



$\vec{A}$ is vector from origin diagonally across left side face

$\vec{A} = (1,0,1) = \hat{x} + \hat{z}$

$\vec{B}$ is vector from origin diagonally across back face

$\vec{B} = (0,1,1) = \hat{y} + \hat{z}$

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$ where θ is angle between $\vec{A}$ and $\vec{B}$

$$\vec{A} \cdot \vec{B} = (\hat{x} + \hat{z}) \cdot (\hat{y} + \hat{z}) = \underbrace{\hat{x} \cdot \hat{y}}_{0} + \underbrace{\hat{x} \cdot \hat{z}}_{0} + \underbrace{\hat{z} \cdot \hat{y}}_{0} + \underbrace{\hat{z} \cdot \hat{z}}_{1} = 1$$

$$\text{or} \quad = (1,0,1) \cdot (0,1,1) = 0 + 0 + 1 = 1$$

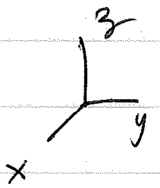
$$|\vec{A}|^2 = \vec{A} \cdot \vec{A} = (1,0,1) \cdot (1,0,1) = 1 + 1 = 2$$

$$|\vec{B}|^2 = \vec{B} \cdot \vec{B} = (0,1,1) \cdot (0,1,1) = 1 + 1 = 2$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

2) Griffiths 1.5

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\times\left[(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\times(C_x\hat{x} + C_y\hat{y} + C_z\hat{z})\right]$$



$$= \vec{A}\times\Big[B_xC_x\underbrace{(\hat{x}\times\hat{x})}_{0} + B_xC_y\underbrace{(\hat{x}\times\hat{y})}_{\hat{z}} + B_xC_z\underbrace{(\hat{x}\times\hat{z})}_{-\hat{y}}$$

$$+ B_yC_x\underbrace{(\hat{y}+\hat{x})}_{-\hat{z}} + B_yC_y\underbrace{(\hat{y}\times\hat{y})}_{0} + B_yC_z\underbrace{(\hat{y}\times\hat{z})}_{\hat{x}}$$

$$+ B_zC_x\underbrace{(\hat{z}\times\hat{x})}_{\hat{y}} + B_zC_y\underbrace{(\hat{z}\times\hat{y})}_{-\hat{x}} + B_zC_z\underbrace{(\hat{z}\times\hat{z})}_{0}\Big]$$

$$= \vec{A}\times\left[(B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + (B_xC_y - B_yC_x)\hat{z}\right]$$

$$= \left[A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right]\times\left[(B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + (B_xC_y - B_yC_x)\hat{z}\right]$$

$$= A_x(B_yC_z - B_zC_y)\underbrace{(\hat{x}\times\hat{x})}_{0} + A_x(B_zC_x - B_xC_z)\underbrace{(\hat{x}\times\hat{y})}_{\hat{z}} + A_x(B_xC_y - B_yC_x)\underbrace{(\hat{x}\times\hat{z})}_{-\hat{y}}$$

$$+ A_y(B_yC_z - B_zC_y)\underbrace{(\hat{y}\times\hat{x})}_{-\hat{z}} + A_y(B_zC_x - B_xC_z)\underbrace{(\hat{y}\times\hat{y})}_{0} + A_y(B_xC_y - B_yC_x)\underbrace{(\hat{y}\times\hat{z})}_{\hat{x}}$$

$$+ A_z(B_yC_z - B_zC_y)\underbrace{(\hat{z}\times\hat{x})}_{\hat{y}} + A_z(B_zC_x - B_xC_z)\underbrace{(\hat{z}\times\hat{y})}_{-\hat{x}} + B_z(B_xC_y - B_yC_x)\underbrace{(\hat{z}\times\hat{z})}_{0}$$

$$\boxed{\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C}) = {} & \left[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\right]\hat{x} \\ & + \left[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)\right]\hat{y} \\ & + \left[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)\right]\hat{z}\end{aligned}}$$

Compare to $\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$= \vec{B}(A_xC_x + A_yC_y + A_zC_z) - \vec{C}(A_xB_x + A_yB_y + A_zB_z)$$

$$= [B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)]\hat{x}$$

cancel

$$+ [B_y(A_xC_x + A_yC_y + A_zC_z) - C_y(A_xB_x + A_yB_y + A_zB_z)]\hat{y}$$

cancel

$$+ [B_z(A_xC_x + A_yC_y + A_zC_z) - C_z(A_xB_x + A_yB_y + A_zB_z)]\hat{z}$$

cancel

$$= [B_xA_yC_y + B_xA_zC_z - C_xA_yB_y - C_xA_zB_z]\hat{x}$$

$$+ [B_yA_xC_x + B_yA_zC_z - C_yA_xB_x - C_yA_zB_z]\hat{y}$$

$$+ [B_zA_xC_x + B_zA_yC_y - C_zA_xB_x - C_zA_yB_y]\hat{z}$$

regroup terms

$$\boxed{\begin{aligned}\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) &= [A_y(B_xC_y - C_xB_y) - A_z(C_xB_z - B_xC_z)]\hat{x}\\ &+ [A_z(B_yC_z - C_yB_z) - A_x(C_yB_x - B_yC_x)]\hat{y}\\ &+ [A_x(B_zC_x - C_zB_x) - A_y(C_zB_y - B_zC_y)]\hat{z}\end{aligned}}$$

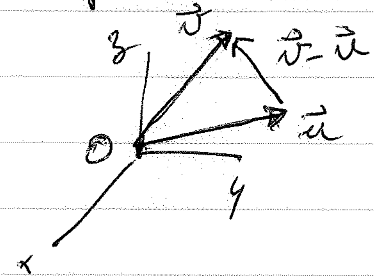
Compare the two boxed expressions and we see they are equal

3) Griffiths 1.7

source $\vec{r}' = (2,8,7) = 2\hat{x} + 8\hat{y} + 7\hat{z}$

field point $\vec{r} = (4,6,8) = 4\hat{x} + 6\hat{y} + 8\hat{z}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ points from source to field point



$\vec{\mathscr{r}} = (4,6,8) - (2,8,7) = (2,-2,1) = 2\hat{x} - 2\hat{y} + \hat{z}$

$|\vec{\mathscr{r}}|^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}} = (2,-2,1)\cdot(2,-2,1) = 4+4+1 = 9$

$|\vec{\mathscr{r}}| = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = 3$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|} = \frac{2\hat{x} - 2\hat{y} + \hat{z}}{3} = \frac{2}{3}\hat{x} - \frac{2}{3}\hat{y} + \frac{1}{3}\hat{z}$$

$$= \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

4) Griffiths 1.13

$$\vec{\mathscr{r}} = (x, y, z) - (x', y', z') = \vec{r} - \vec{r}\,'$$
$$= (x - x', y - y', z - z')$$
$$\mathscr{r} = |\vec{\mathscr{r}}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

a) $$\vec{\nabla}(\mathscr{r}^2) = \vec{\nabla}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)$$

$$= \hat{x}\frac{\partial}{\partial x}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)$$
$$+ \hat{y}\frac{\partial}{\partial y}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)$$
$$+ \hat{z}\frac{\partial}{\partial z}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)$$

$$= \hat{x}\left[2(x-x')\right] + \hat{y}\left[2(y-y')\right] + \hat{z}\left[2(z-z')\right]$$

$$= 2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}$$
$$= 2(\vec{r} - \vec{r}\,')$$
$$= 2\vec{\mathscr{r}}$$

b) $$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \hat{x}\,\frac{\partial}{\partial x}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right)$$

$$+\hat{y}\,\frac{\partial}{\partial y}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right)$$

$$+\hat{z}\,\frac{\partial}{\partial z}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right)$$

$$= \hat{x}\left(\frac{(-\frac{1}{2})(2)(x-x')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right)$$

$$+\hat{y}\left(\frac{(-\frac{1}{2})(2)(y-y')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right)$$

$$+\hat{z}\left(\frac{(-\frac{1}{2})(2)(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\right)$$

$$= -\frac{\hat{x}\,(x-x')}{\mathscr{r}^3} - \frac{\hat{y}\,(y-y')}{\mathscr{r}^3} - \frac{\hat{z}\,(z-z')}{\mathscr{r}^3}$$

$$= -\left(\frac{\vec{r}-\vec{r}\,'}{\mathscr{r}^3}\right) = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{as } \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$

c) $\vec{\nabla}(\mathscr{r}^n) = \hat{x}\frac{\partial}{\partial x}\left([(x-x')^2+(y-y')^2+(z-z')^2]^{n/2}\right)$

$+\hat{y}\frac{\partial}{\partial y}\left([(x-x')^2+(y-y')^2+(z-z')^2]^{n/2}\right)$

$+\hat{z}\frac{\partial}{\partial z}\left([(x-x')^2+(y-y')^2+(z-z')^2]^{n/2}\right)$

$= \hat{x}\left(\left(\frac{n}{2}\right)[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}(2)(x-x')\right)$

$+\hat{y}\left(\left(\frac{n}{2}\right)[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}(2)(y-y')\right)$

$+\hat{z}\left(\left(\frac{n}{2}\right)[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n}{2}-1}(2)(z-z')\right)$

$= n[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{n-2}{2}}\left((x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}\right)$

$= n\mathscr{r}^{n-2}(\vec{r}-\vec{r}\,') = n\mathscr{r}^{n-2}\vec{\mathscr{r}} = n\mathscr{r}^{n-2}\mathscr{r}\hat{\mathscr{r}}$

using $\mathscr{r}\hat{\mathscr{r}} = \vec{\mathscr{r}}$

$= n\mathscr{r}^{n-1}\hat{\mathscr{r}}$

Check parts (a) and (b)

$$\left.\begin{aligned} n=2 \quad & \vec{\nabla}(\mathscr{r}^2) = 2\mathscr{r}^1\hat{\mathscr{r}} = 2\vec{\mathscr{r}} \\ n=-1 \quad & \vec{\nabla}(\mathscr{r}^{-1}) = -1\mathscr{r}^{-2}\hat{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \end{aligned}\right\}\text{ agrees!}$$