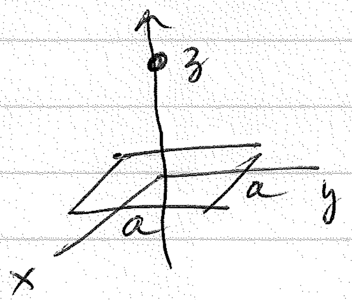
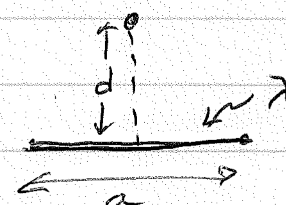
1) Field along z axis above square loop of side a with uniform line charge λ



From lecture we saw that the $\vec{E}$ field, at a point directly above the midpoint of a line of length a with uniform λ is

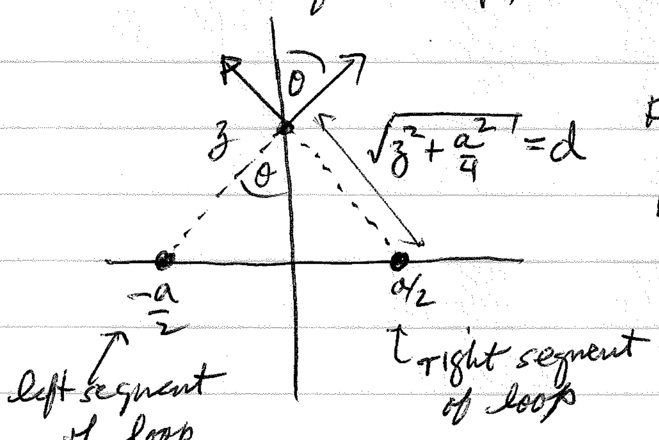


compare to Ex 2.2
with $L \to \frac{a}{2}$
$z \to d$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{\lambda a}{d\sqrt{d^2 + \left(\frac{a}{2}\right)^2}}$$

$\vec{E}$ points in direction from midpoint of line to observer

For the square loop, consider the y-z plane



Field from each of the two segments point as shown. We see the components $\perp z$ will cancell, while the $\hat{z}$ components add. So the net field above the square loop is in $\hat{z}$ direction, and equal to 4 times the z-component of field from one side of the square

$$\cos\theta = \frac{z}{\sqrt{z^2 + \frac{a^2}{4}}}$$

The z component of $\vec{E}$ from one side of square has magnitude

$$\frac{1}{4\pi\varepsilon_0} \frac{\lambda a}{d\sqrt{d^2 + \frac{a^2}{4}}} \underbrace{\cos\theta}_{\text{project on } \hat{z} \text{ axis}}$$

here the distance from wire to observer is

$$d = \sqrt{z^2 + \frac{a^2}{4}}$$

and $\cos\theta = \dfrac{z}{\sqrt{z^2 + \frac{a^2}{4}}}$

So

$$\vec{E}(z\hat{z}) = \hat{z}\, 4 \frac{1}{4\pi\varepsilon_0} \frac{\lambda a}{\sqrt{z^2 + \frac{a^2}{4}} \cdot \sqrt{z^2 + \frac{a^2}{4} + \frac{a^2}{4}}} \cdot \frac{z}{\sqrt{z^2 + \frac{a^2}{4}}}$$

$$= \hat{z} \frac{\lambda a}{\pi \varepsilon_0} \frac{z}{\left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}$$

Note: E(z) = -E(-z) is antisymmetric as one would expect

<u>Check</u>: at $z = 0$, $\vec{E} = 0$

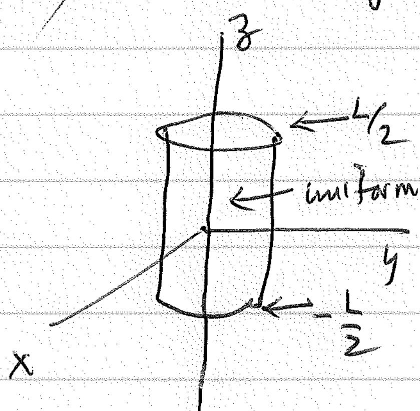
For $z \gg a$, $z^2 + \frac{a^2}{4} \simeq z^2 + \frac{a^2}{2} \approx z^2$

$$\vec{E} \simeq \hat{z} \frac{\lambda a}{\pi \varepsilon_0} \frac{z}{z^2 \cdot z} = \hat{z} \frac{4\lambda a}{4\pi\varepsilon_0} \frac{1}{z^2}$$

Total charge is $Q = 4a\lambda$ so

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{z^2} \hat{z}$$ as expected.

2) Field along axis of cylindrical shell of charge of radius R, length L



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{surface}} da'\,\sigma\,\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} dz' \int_0^{2\pi} d\varphi\, R\, \frac{(z-z')\hat{z} - R\cos\varphi\,\hat{x} - R\sin\varphi\,\hat{y}}{\left[(z-z')^2 + R^2\right]^{3/2}}$$

$\vec{r} = z\hat{z}$

$\vec{r}\,' = z'\hat{z} + R(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y})$

when integrate over φ, $\int_0^{2\pi} d\varphi\cos\varphi = \int_0^{2\pi} d\varphi\sin\varphi = 0$

so only $\hat{z}$ component does not vanish

$$\vec{E}(z\hat{z}) = \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} dz'\,\frac{(z-z')}{\left[(z-z')^2+R^2\right]^{3/2}}$$

Change integration variable to $\bar{z} = z - z'$ $\quad d\bar{z} = -dz'$

$z' = (-\frac{L}{2}, \frac{L}{2}) \rightarrow \bar{z} = (z+\frac{L}{2}, z-\frac{L}{2})$ flip limits of integration gives (−)

$$\vec{E}(z\hat{z}) = \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0}\int_{z-\frac{L}{2}}^{z+\frac{L}{2}} d\bar{z}\,\frac{\bar{z}}{(\bar{z}^2+R^2)^{3/2}}$$

$$= \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0}\left(\frac{-1}{(\bar{z}^2+R^2)^{1/2}}\right)\Bigg|_{z-\frac{L}{2}}^{z+\frac{L}{2}}$$

Note: E(z) = -E(-z) is antisymmetric as one would expect

$$\boxed{\vec{E}(z\hat{z}) = \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(z-\frac{L}{2})^2+R^2}} - \frac{1}{\sqrt{(z+\frac{L}{2})^2+R^2}}\right]}$$

Check limits: at $z=0$, $\vec{E}=0$ as expected by symmetry

What if $z \gg L$, $z \gg R$? Find $\vec{E}$ to lowest non zero order in $\frac{L}{z}$ and $\frac{R}{z}$.

$$\frac{1}{\sqrt{(z-\frac{L}{2})^2+R^2}} = \frac{1}{z}\frac{1}{\sqrt{(1-\frac{L}{2z})^2+\frac{R^2}{z^2}}} = \frac{1}{z\sqrt{1-\frac{L}{z}+\frac{L^2}{4z^2}+\frac{R^2}{z^2}}}$$

to lowest order in small quantities $\frac{L}{z}$, $\frac{R}{z}$

$$\simeq \frac{1}{z}\frac{1}{\left(1-\frac{L}{2z}\right)} \simeq \frac{1}{z}\left(1+\frac{L}{2z}\right)$$ ignore $\frac{L^2}{z^2}$ and $\frac{R^2}{z^2}$ terms

similarly

$$\frac{1}{\sqrt{(z+\frac{L}{2})^2+R^2}} \simeq \frac{1}{z}\left(1-\frac{L}{2z}\right)$$

So $$\vec{E}(z\hat{z}) \simeq \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0}\,\frac{1}{z}\left[\left(1+\frac{L}{2z}\right)-\left(1-\frac{L}{2z}\right)\right]$$

$$\simeq \hat{z}\,\frac{2\pi\sigma R}{4\pi\varepsilon_0 z}\left[\frac{L}{z}\right] = \hat{z}\,\frac{2\pi R L\sigma}{4\pi\varepsilon_0 z^2}$$

Total charge on cylindrical shell is σ (area) $= 2\pi R L\sigma = Q$

$$\vec{E}(z\hat{z}) = \frac{Q}{4\pi\varepsilon_0 z^2}\,\hat{z}$$ just like point charge Q when $\vec{r}$ is far away!