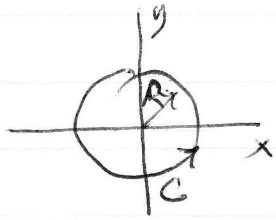
1.55 $\quad v = ay\,\hat{x} + bx\,\hat{y}$




C is circular path of radius R centered at origin in xy plane

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{v}) = \oint_C d\vec{l}\cdot\vec{v}$$

Do the left hand side first

$$\vec{\nabla}\times\vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= \hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(b-a)$$

Integrate over the area of the circle in the xy plane

$$\int_0^R dr\,r \int_0^{2\pi} d\varphi\ \hat{z}\cdot(\vec{\nabla}\times\vec{v}) \qquad \text{using } d\vec{a} = da\,\hat{z} \qquad da = dr\,r\,d\varphi$$

$$= \int_0^R dr \int_0^{2\pi} d\varphi\ (b-a) = (b-a)\pi R^2$$

Now do the right hand side

Now do the line integral $\qquad$ take $d\vec{l} = R\,d\varphi\,\hat{\varphi}$

$$\oint_C d\vec{l}\cdot\vec{v} = \int_0^{2\pi} d\varphi\, R\,\hat{\varphi}\cdot\vec{v}$$

Note: direction $\hat{\varphi}$ depends on angle $\varphi$; we need to convert $\hat{\varphi}$ to Cartesian coordinates to do the integral use from previous homework

$$\hat{\varphi} = \cos\varphi\,\hat{y} - \sin\varphi\,\hat{x}$$

$$\oint_C d\vec{l}\cdot\vec{v} = \int_0^{2\pi} d\varphi\, R\left(\cos\varphi\, v_y - \sin\varphi\, v_x\right)$$

$$= \int_0^{2\pi} d\varphi\, R\left(\cos\varphi\, bx - \sin\varphi\, ay\right)$$

convert x and y to cylindrical coords
$x = R\cos\varphi\,,\quad y = R\sin\varphi$

$$= \int_0^{2\pi} d\varphi\, R^2\left(b\cos^2\varphi - a\sin^2\varphi\right) = \pi R^2(b-a)$$

since $\int_0^{2\pi} d\varphi\cos^2\varphi = \int_0^{2\pi} d\varphi\sin^2\varphi = \pi$

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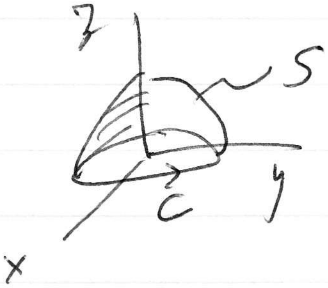
For fun, we could also do $\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{v})$ over the surface of a hemisphere for which C is the equator




now $d\vec{a} = \underbrace{R^2\sin\theta\, d\theta\, d\varphi}\,\hat{r}$

differential area on surface of sphere

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{v}) = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, \sin\theta\, R^2\, \hat{r}\cdot(\vec{\nabla}\times\vec{v})$$

upper limit is $\pi/2$ since integrate over hemisphere

$$\hat{r} \cdot (\vec{\nabla} \times \vec{v}) = \hat{r} \cdot \hat{z}\,(b-a) = (b-a)\cos\theta$$

$$\int d\vec{a} \cdot (\vec{\nabla} \times \vec{v}) = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \; R^2 \sin\theta \; (b-a)\cos\theta$$

$$= 2\pi R^2 (b-a) \underbrace{\int_0^{\pi/2} d\theta \; \sin\theta\cos\theta}_{\left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2} = \frac{1}{2}}$$

$$= 2\pi R^2 (b-a)\,\frac{1}{2} \quad = \pi R^2 (b-a)$$

same answer as before!

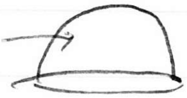
Surface integral of $\vec{\nabla} \times \vec{v}$ is the same over <u>all</u> surfaces that have the same bounding curve C

Prob 1.62 (a) – (d)

$$\vec{a} \equiv \int_S d\vec{a} \quad \text{where } d\vec{a} = da\, \hat{n}$$

For a planar surface, then $\vec{a} = a\hat{n}$ with $a$ the area, $\hat{n}$ the outward normal

a) For hemisphere at radius R



$$\vec{a} = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, R^2 \sin\theta\, \hat{r} \qquad \text{outward normal is } \hat{r}$$

$$= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, R^2 \sin\theta \underbrace{\left(\cos\theta\, \hat{z} + \sin\theta\cos\varphi\, \hat{x} + \sin\theta\sin\varphi\, \hat{y}\right)}_{\hat{r}}$$

do the $\varphi$-integrations

$$\int_0^{2\pi} d\varphi \cos\varphi = \int_0^{2\pi} d\varphi \sin\varphi = 0 \qquad \int_0^{2\pi} d\varphi = 2\pi$$

so the $\hat{x}$ and $\hat{y}$ components vanish

$$\vec{a} = 2\pi R^2 \int_0^{\pi/2} d\theta \sin\theta\cos\theta\, \hat{z}$$

$$= 2\pi R^2 \left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2} \hat{z} = 2\pi R^2 \left(\tfrac{1}{2}\right) \hat{z}$$

$$\boxed{\vec{a} = \pi R^2\, \hat{z}}$$

b) Show that $\vec{a} = 0$ for any closed surface.

Use divergence theorem $\quad \oint_S d\vec{a} \cdot \vec{v} = \int_V d^3r \; \vec{\nabla} \cdot \vec{v}$

$$a_x = \oint_S d\vec{a} \cdot \hat{x} = \int_V d^3r \, (\vec{\nabla} \cdot \hat{x}) = 0 \quad \text{as } \hat{x} \text{ is a constant}$$

$$a_y = \oint_S d\vec{a} \cdot \hat{y} = \int_V d^3r \, (\vec{\nabla} \cdot \hat{y}) = 0 \quad \text{as } \hat{y} \text{ is a constant}$$

$$a_z = \oint_S d\vec{a} \cdot \hat{z} = \int_V d^3r \, (\nabla \cdot \hat{z}) = 0 \quad \text{as } \hat{z} \text{ is a constant}$$

Thus $\oint_S d\vec{a} = 0$ for any closed surface $S$

c) Consider two surfaces $S_1$ and $S_2$ that have the same boundary curve $C$.

For example


$S_1$ is planar area of circle
$S_2$ is hemispherical area

Let $S_1 + S_2$ be the closed surface
Then

$$\oint_{S_1+S_2} d\vec{a} = 0 = \int_{S_1} d\vec{a} - \int_{S_2} d\vec{a}$$

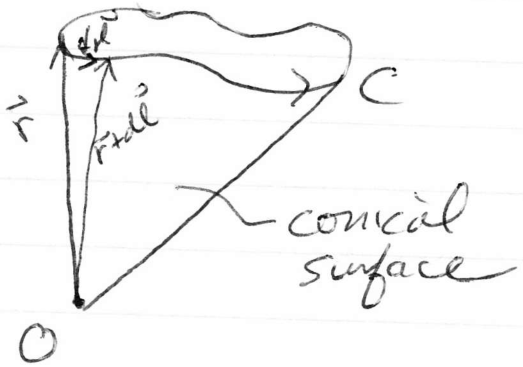
minus sign comes because when we regard $S_2$ as surface bounded by $C$, the outward normal $\hat{n}$ has opposite sign as when we regard $S_2$ as closing up $S_1$

$\Rightarrow \int_{S_1} d\vec{a} = \int_{S_2} d\vec{a}$ has same value for any surface S that has same bounding curve C.

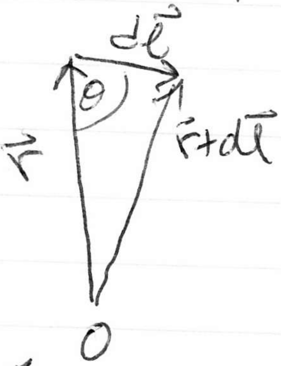
d) For surface S with bounding curve C, show that

$$\int_S d\vec{a} = \frac{1}{2}\oint_C \vec{r} \times d\vec{l}$$

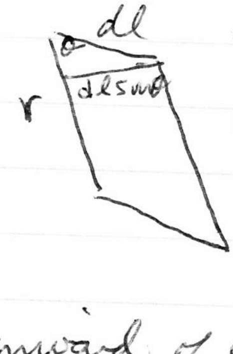
Since we know the integral has the same value for any surface S that is bounded by C, so choose the surface formed by the cone subtended by C with vertex at the origin




Consider the triangular segment of this cone between $\vec{r}$ and $\vec{r}+d\vec{l}$




Now $|\vec{r} \times d\vec{l}| = r\,dl\sin\theta$




= area of parallelogram

$\frac{1}{2} r\,dl\sin\theta$ = area of triangular wedge.

direction of $\vec{r} \times d\vec{l}$ is pointing inward of cone but this is the correct direction if we choose $\hat{n}$ consistent with right hand rule for $d\vec{l}$ as shown

so $\frac{1}{2}\vec{r} \times d\vec{l} = da\,\hat{n}$ $\quad da$ = area of triangular wedge

$$\Rightarrow \frac{1}{2}\oint_C \vec{r} \times d\vec{l} = \int_{S_{cone}} da\,\hat{n} = \int_{S \text{ any surface}} d\vec{a}$$