$\boxed{1.50}$

a) $\vec{F}_1(\vec{r}) = x^2 \hat{z}$ $\qquad \vec{F}_2(\vec{r}) = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\cdot\vec{F}_1 = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 0$$

$$\vec{\nabla}\times\vec{F}_1 = \hat{x}\left(\frac{\partial F_{1z}}{\partial y} - \frac{\partial F_{1y}}{\partial z}\right) + \hat{y}\left(\frac{\partial F_{1x}}{\partial z} - \frac{\partial F_{1z}}{\partial x}\right) + \hat{z}\left(\frac{\partial F_{1y}}{\partial x} - \frac{\partial F_{1x}}{\partial y}\right)$$

$$= 0 + \left(-2x\hat{y}\right) + 0 = -2x\hat{y}$$

$$\vec{\nabla}\cdot\vec{F}_2 = \frac{\partial F_{2x}}{\partial x} + \frac{\partial F_{2y}}{\partial y} + \frac{\partial F_{2z}}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\vec{\nabla}\times\vec{F}_2 = 0$$

so we can write $\vec{F}_1 = \vec{\nabla}\times\vec{W}$ and $\vec{F}_2 = -\vec{\nabla}u$

To find $\vec{W}$ we want $\dfrac{\partial W_y}{\partial x} - \dfrac{\partial W_x}{\partial y} = x^2$

choose $\boxed{\vec{W} = \frac{1}{3}x^3\hat{y}}$ then $\vec{\nabla}\times\vec{W} = \hat{x}\left(\dfrac{\partial W_z}{\partial y} - \dfrac{\partial W_y}{\partial z}\right)$

$$+\hat{y}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) + \hat{z}\left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y}\right)$$

$$= 0 + 0 + \hat{z}\left(x^2 - 0\right) = x^2\hat{z} = \vec{F}_1$$

To find $U(\vec{r})$ we can take

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{l} \cdot \vec{F}_2$$

where it does not matter what we choose for the point $\vec{r}_0$, or the path we take from $\vec{r}_0$ to $\vec{r}$ (since $\vec{\nabla} \times \vec{F}_2 = 0$).

Choose $\vec{r}_0 = 0$, and choose the path to go in a straight line from the origin to the point $\vec{r}$.

We can parameterize the path as $\vec{r}\,'(t) = (xt)\hat{x} + (yt)\hat{y} + (zt)\hat{z}$ with t going from 0 to 1

Then $$U(\vec{r}) = -\int_0^1 dt \, \frac{d\vec{r}\,'(t)}{dt} \cdot \vec{F}_2(\vec{r}\,'(t))$$

$$= -\int_0^1 dt \, (x\hat{x} + y\hat{y} + z\hat{z}) \cdot (xt\,\hat{x} + yt\,\hat{y} + zt\,\hat{z})$$

$$= -\int_0^1 dt \, (x^2+y^2+z^2)t = -\tfrac{1}{2}(x^2+y^2+z^2)$$

So $\boxed{U(\vec{r}) = -\tfrac{1}{2} r^2}$ since $\int_0^1 dt\, t = \frac{1}{2}$

Check $$-\vec{\nabla}\left(-\tfrac{1}{2}r^2\right) = \tfrac{1}{2}\vec{\nabla}(x^2+y^2+z^2)$$

$$= \tfrac{1}{2}(2x\,\hat{x} + 2y\,\hat{y} + 2z\,\hat{z})$$

$$= x\hat{x} + y\hat{y} + z\hat{z} = \vec{F}_2$$

b) $\vec{F}_3 = yz\,\hat{x} + zx\,\hat{y} + xy\,\hat{z}$

Show we can write both $\vec{F}_3 = -\vec{\nabla} U$ and $\vec{F}_3 = \vec{\nabla} \times \vec{W}$

$$\vec{\nabla} \cdot \vec{F}_3 = \frac{\partial F_{3x}}{\partial x} + \frac{\partial F_{3y}}{\partial y} + \frac{\partial F_{3z}}{\partial z} = 0$$

$$\vec{\nabla} \times \vec{F}_3 = \hat{x}\left(\frac{\partial F_{3z}}{\partial y} - \frac{\partial F_{3y}}{\partial z}\right) + \hat{y}\left(\frac{\partial F_{3x}}{\partial z} - \frac{\partial F_{3z}}{\partial x}\right) + \hat{z}\left(\frac{\partial F_{3y}}{\partial x} - \frac{\partial F_{3x}}{\partial y}\right)$$

$$= \hat{x}(x - x) + \hat{y}(y - y) + \hat{z}(z - z) = 0$$

since $\vec{\nabla} \cdot \vec{F}_3 = 0$ we can write $\vec{F}_3 = \vec{\nabla} \times \vec{W}$

since $\vec{\nabla} \times \vec{F}_3 = 0$ we can write $\vec{F}_3 = -\vec{\nabla} U$

To find $\vec{W}$ we want

$$\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} = F_{3x} = yz$$

$$\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} = F_{3y} = zx$$

$$\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} = F_{3z} = xy$$

Try $W_z = \frac{1}{2} y^2 z$, $W_x = \frac{1}{2} z^2 x$, $W_y = \frac{1}{2} x^2 y$

$$\boxed{\vec{W} = \frac{1}{2}\left(z^2 x\,\hat{x} + x^2 y\,\hat{y} + y^2 z\,\hat{z}\right)}$$

Can verify that $\vec{\nabla} \times \vec{W} = (yz)\,\hat{x} + (zx)\,\hat{y} + (xy)\,\hat{z} = \vec{F}_3$

To find U we can do $U(\vec{r}) = -\int_{r_0}^{\vec{r}} d\vec{l} \cdot \vec{F}_3$ or can try a more direct approach

$$\frac{\partial U}{\partial x} = -F_{3x} = yz$$

$$\frac{\partial U}{\partial y} = -F_{3y} = zx$$

$$\frac{\partial U}{\partial z} = -F_{3z} = xy$$

Try $\boxed{U(\vec{r}) = -xyz}$ can verify that $-\vec{\nabla} U = \vec{F}_3$

2-9) $\vec{E}(\vec{r}) = k r^3 \hat{r}$ in spherical coordinates

a) Find charge density $\rho(\vec{r})$

We know $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$

evaluate $\vec{\nabla} \cdot \vec{E}$ in spherical coordinates. Since $\vec{E}$ points in radial direction and depends only on the radial coordinate, only the radial term in $\vec{\nabla} \cdot \vec{E}$ is not zero.

$$\text{So } \vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 E(r)\right) = \frac{k}{r^2} \frac{\partial}{\partial r}\left(r^5\right)$$

$$= \frac{5k}{r^2} r^4 = 5kr^2 = \rho/\varepsilon_0$$

$$\boxed{\rho(\vec{r}) = 5\varepsilon_0 k r^2}$$

b) Charge within sphere of radius R centered at origin

$$\text{i) } Q = \underbrace{\int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi}_{\text{volume integral in spherical coordinates}} \; \rho(\vec{r})$$

$$= 4\pi \int_0^R dr\, r^2 \rho(r) = (4\pi)(5\varepsilon_0 k) \int_0^R dr\, r^2 (r^2)$$

$$= (4\pi)(5\varepsilon_0 k) \frac{R^5}{5} = \boxed{4\pi \varepsilon_0 k R^5 = Q}$$

or from Gauss' Law

2) $\frac{Q}{\varepsilon_0} = \oint_S d\vec{a} \cdot \vec{E}$ where S is the surface of the sphere of radius R.

$= \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta R^2 \, \hat{r} \cdot (kR^3 \hat{r})$

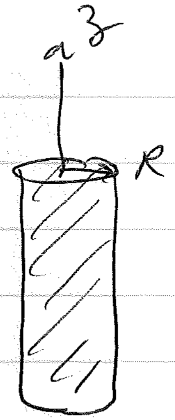
since evaluating $\vec{E}$ at $r = R$

$= (4\pi) k R^5$

from angular integrations

$$\boxed{Q = 4\pi \varepsilon_0 k R^5}$$

2.13/ but more general



uniform charge density ρ inside infinitely long cylindrical wire of radius R. Charge per unit length is λ

so $\lambda = \pi R^2 \rho$

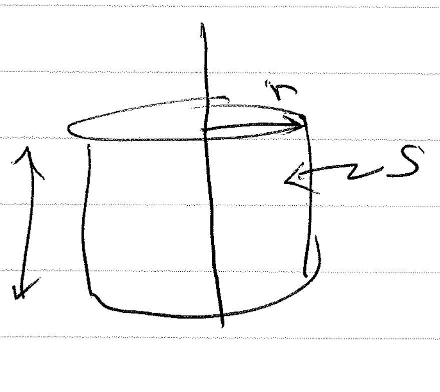
Find $\vec{E}$ inside and outside the cylinder using Gauss' law.

1) Integral method

~~By symmetry of rotational invariance about z, $\vec{E}(\vec{r})$ must have the form $E(r)\hat{r}$ where r is the cylindrical radial direction~~

Because of rotational symmetry about the z axis, $\vec{E}$ cannot involve polar angle φ. Because of translational symmetry along $\hat{z}$ axis, $\vec{E}$ cannot involve the coordinate z.

$\Rightarrow \vec{E}(\vec{r})$ must have the form $E(r)\hat{r}$ where here r is the cylindrical radial direction, $\hat{r}$ is the corresponding cylindrical radial unit vector



evaluate $\oint_S d\vec{a}\cdot\vec{E}$ on a cylindrical surface of radius r and length L.

Since $\vec{E}$ is in the $\hat{r}$ direction there is no ~~...~~ contribution from top and bottom surfaces, since $\hat{z}\cdot\vec{E}=0$

$$\oint_S d\vec{a}\cdot\vec{E} = \int_0^L dz \int_0^{2\pi} d\varphi \; r \; \hat{r}\cdot\vec{E} = L\,2\pi\, r\, E(r) = \frac{Q_{encl}}{\varepsilon_0}$$

Now if $r > R$, then $Q_{encl} = L\lambda$ so

$$\boxed{\vec{E}(\vec{r}) = \frac{\lambda \;\hat{n}}{2\pi\varepsilon_0\, r} \qquad r > R}$$

If $r < R$ then

$$Q_{encl} = \int_0^L dz \int_0^{2\pi} d\varphi \int_0^r dr'\, r' \rho = (L)(2\pi)\frac{r^2}{2}\rho = L\pi r^2 \rho$$

$$= \frac{L\lambda r^2}{R^2} \qquad \text{using } \lambda = \pi R^2 \rho$$

so $$L\,2\pi\, r\, E(r) = \frac{L\lambda r^2}{\varepsilon_0 R^2}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{r}{R^2}\,\hat{r}} \qquad r < R$$

2) Solve using Gauss' Law in differential form

$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$ with $\vec{E}(\vec{r}) = E(r)\hat{r}$

evaluate divergence in cylindrical coordinates

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE) = \frac{\rho(r)}{\varepsilon_0} \qquad \rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}$$

$$\frac{\partial}{\partial r}(rE) = \frac{r\rho(r)}{\varepsilon_0}$$

integrate from $r' = 0$ to $r' = r$

$$rE(r) - 0 = \frac{1}{\varepsilon_0}\int_0^r dr'\, r'\rho(r')$$

$$= \begin{cases} \frac{1}{\varepsilon_0}\frac{1}{2}R^2\rho & \text{for } r > R \\ \frac{1}{\varepsilon_0}\frac{1}{2}r^2\rho & \text{for } r < R \end{cases}$$

$$E(r) = \begin{cases} \frac{R^2\rho}{2\varepsilon_0 r} & r > R \\ \frac{r\rho}{2\varepsilon_0} & r < R \end{cases}$$

$\lambda = \pi R^2 \rho$ so $\rho = \frac{\lambda}{\pi R^2}$ substitute in

$$\boxed{\vec{E}(\vec{r}) = \begin{cases} \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} & r > R \\ \frac{\lambda}{2\pi\varepsilon_0}\frac{r}{R^2}\hat{r} & r < R \end{cases}}$$