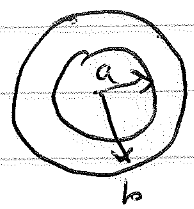
2.15




Thick spherical shell, inner radius $a$, outer radius $b$

$\rho(r) = \frac{k}{r^2} \qquad a \le r \le b$ depends only on radial coord

Find electric field $\vec{E}$.

By symmetry $\vec{E}(\vec{r}) = E(r)\hat{r}$ – depends only on radial coord, points in radial direction

Gauss' law: $\oint_S d\vec{a}\cdot\vec{E} = \frac{Q_{encl}}{\varepsilon_0}$

choose as the surface $S$ the surface of a sphere of radius $r$

$$\oint_S d\vec{a}\cdot\vec{E} = \oint da\, \hat{r}\cdot\hat{r}\, E(r) = 4\pi r^2 E(r) = \frac{Q_{encl}}{\varepsilon_0}$$

for $r < a$, $Q_{encl} = 0$ so $E = 0$

for $r > b$, $Q_{encl} = \int_a^b dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta\, r^2 \rho(r)$

$$= 4\pi \int_a^b dr\, r^2 \frac{k}{r^2} = 4\pi k (b-a)$$

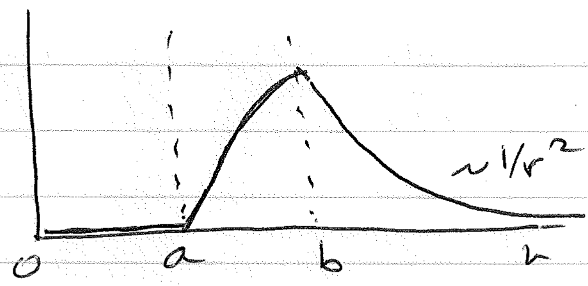
for $a < r < b$, $Q_{encl} = \int_0^r dr' \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sin\theta\, r'^2 E(r')$

$$= 4\pi k (r-a)$$

So

$$\vec{E}(\vec{r}) = \begin{cases} 0 & 0 < r < a \\ \frac{k(r-a)}{\varepsilon_0 r^2}\hat{r} & a < r < b \\ \frac{k(b-a)}{\varepsilon_0 r^2}\hat{r} & b < r \end{cases}$$

Sketch $E(r)$




2.23) Now find $V(\vec{r})$ for the same geometry

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} d\vec{l} \cdot E(\vec{r})$$ by symmetry $V$ depends only on $|\vec{r}|$

↖ use $\infty$ as reference point so that $V(\infty) = 0$

choose path to be in radial direction from $\infty$ to $r$

$$V(r) = -\int_{\infty}^{r} d\vec{l}' \cdot \vec{E}(\vec{r}') = \int_{r}^{\infty} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

$$= \int_{r}^{\infty} dr' \, \hat{r} \cdot \hat{r} \, E(r') = \int_{r}^{\infty} dr' \, E(r')$$

For $r > b$ $\quad V(r) = \int_{r}^{\infty} dr' \, \frac{k(b-a)}{\varepsilon_0 r'^2} = \left( \frac{-k(b-a)}{\varepsilon_0 r'} \right)_{r}^{\infty}$

$$\boxed{V(r) = \frac{k(b-a)}{\varepsilon_0 r} \qquad b < r}$$

For $a < r < b$ $\quad V(r) = \int_{r}^{\infty} dr' E(r') = \int_{r}^{b} dr' \, E(r')$

$$+ \int_{b}^{\infty} dr' \, E(r')$$

$$= \int_{r}^{b} dr' \, \frac{k(r'-a)}{\varepsilon_0 r'^2} + \frac{k(b-a)}{\varepsilon_0 b}$$

$$= \int_r^b dr' \frac{k}{\varepsilon_0 r'} - \int_r^b dr' \frac{ka}{\varepsilon_0 r'^2} + \frac{k(b-a)}{\varepsilon_0 b}$$

$$= \left(\frac{k}{\varepsilon_0} \ln r'\right)\Big|_r^b - \left(\frac{-ka}{\varepsilon_0 r'}\right)\Big|_r^b + \frac{k(b-a)}{\varepsilon_0 b}$$

$$= \frac{k}{\varepsilon_0} \ln(b/r) + \frac{ka}{\varepsilon_0 b} - \frac{ka}{\varepsilon_0 a} + \frac{k(b-a)}{\varepsilon_0 b}$$

$$= \frac{k}{\varepsilon_0} \ln(b/r) + \frac{ka}{\varepsilon_0 b} - \frac{k}{\varepsilon_0} + \frac{k}{\varepsilon_0} - \frac{ka}{\varepsilon_0 b}$$

$$\boxed{V(r) = \frac{k}{\varepsilon_0} \ln(b/r) \qquad a<r<b}$$

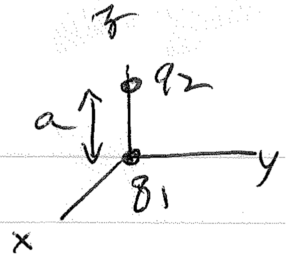
For $r<a$ $\quad V(r) = \int_r^a dr' E(r') + V(a)$

but $E=0$ for $r<a$, so $V(r) = V(a)$ for all $r<a$

$$\boxed{V(r) = \frac{k}{\varepsilon_0} \ln(b/a) \qquad r<a}$$

So at origin $r=0$, $\boxed{V(0) = \frac{k}{\varepsilon_0} \ln(b/a)}$

2.37




Compute energy of two charges using $W = \varepsilon_0 \int d^3r\, \vec{E}_1 \cdot \vec{E}_2$

$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \vec{E}_2 = \frac{q_2}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}$$

with $\vec{r}_2 = a\hat{z}$ location of $q_2$

use spherical coords $\vec{r} = r\hat{r}$

$$\hat{r}\cdot\vec{r} = r, \quad \hat{r}\cdot\vec{r}_2 = a\,\hat{r}\cdot\hat{z} = a\cos\theta$$

$$|\vec{r}-\vec{r}_2| = (r^2+a^2-2\vec{r}\cdot a\hat{z})^{1/2} = (r^2+a^2-2ra\cos\theta)^{1/2}$$

$$d^3r = dr\,d\theta\,d\varphi\,\sin\theta\, r^2$$

$$\vec{E}_1\cdot\vec{E}_2 = \frac{q_1 q_2}{(4\pi\varepsilon_0)^2}\frac{(r-a\cos\theta)}{r^2(r^2+a^2-2ra\cos\theta)^{3/2}}$$

$$W = \varepsilon_0\int d^3r\,\vec{E}_1\cdot\vec{E}_2 = \frac{\varepsilon_0}{(4\pi\varepsilon_0)^2}q_1q_2\int_0^\infty dr\int_0^\pi d\theta\int_0^{2\pi} d\varphi\,\sin\theta\,\frac{r^2\,(r-a\cos\theta)}{r^2(r^2+a^2-2ra\cos\theta)^{3/2}}$$

do integral over $\varphi$

$$= \frac{2\pi\varepsilon_0}{(4\pi\varepsilon_0)^2}q_1q_2\int_0^\pi d\theta\,\sin\theta\int_0^\infty dr\,\frac{(r-a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{3/2}}$$

use $$\frac{(r-a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{3/2}} = \frac{d}{dr}\left[\frac{-1}{(r^2+a^2-2ra\cos\theta)^{1/2}}\right]$$

$$W = \frac{1}{8\pi\varepsilon_0}q_1q_2\int_0^\pi d\theta\sin\theta\left[\frac{-1}{(r^2+a^2-2ra\cos\theta)^{1/2}}\right]_0^\infty$$

$$= \frac{1}{8\pi\varepsilon_0}q_1q_2\int_0^\pi d\theta\sin\theta\,\frac{1}{\sqrt{a^2}} = \frac{q_1q_2}{8\pi\varepsilon_0 a}\int_0^\pi d\theta\sin\theta$$

$$\int_0^{\pi} d\theta \sin\theta = \left(-\cos\theta\right)_0^{\pi} = 2$$

$$\boxed{W = \frac{q_1 q_2}{4\pi\varepsilon_0 a}}$$

This is exactly the same as one gets from

$$W = \frac{q_1 q_2}{4\pi\varepsilon_0 |\bar{r}_1 - \bar{r}_2|}$$