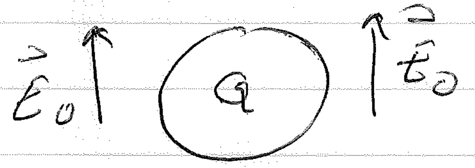
1) 3.21) Potential outside conducting sphere with charge Q, radius R, in uniform external $\vec{E}_0 = E_0 \hat{z}$.



We already found the solution for a grounded sphere in an external $\vec{E}_0$

$$V_{grounded}(\vec{r}) = -E_0 \left(r - \frac{R^3}{r^2}\right) \cos\theta$$

For $r = R$ we set $V_{grounded}(R,\theta) = 0$ as we must

For the sphere with charge Q, let us continue to set $V=0$ on the sphere. Because we can always add an arbitrary constant to V, we have the freedom to make this choice.

The above grounded sphere has net charge $= 0$ since the induced charge is $\sigma = 3\epsilon_0 E_0 \cos(\theta)$ integrates to zero.

So for the charged sphere we want to add charge Q in such a way that the net $\vec{E}$ field stays $\perp$ surface of sphere, and $V=0$.

We can do this by adding a uniform charge density $\sigma_0 = \frac{Q}{4\pi R^2}$ on surface of sphere.

This would give zero E field inside the sphere, and an E that is perpendicular to the surface.

The potential V_Q from such a distribution is

$$V_Q(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r} + C$$

where we can choose the constant C so that the potential V vanishes at our desired reference point. Since we want $V(R,\theta) = 0$, then

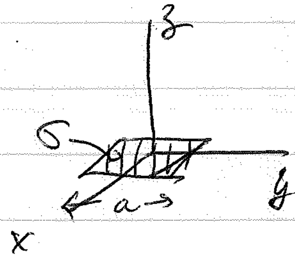
$$C = -\frac{Q}{4\pi\varepsilon_0 R}$$

So finally we have for the charged sphere in the external $\vec{E}_0$

$$V(r,\theta) = V_{grounded}(r,\theta) + V_Q(r,\theta)$$

$$\boxed{V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{Q}{4\pi\varepsilon_0 r} - \frac{Q}{4\pi\varepsilon_0 R}}$$

2) Square of length a in xy plane at $z=0$ centered at origin with uniform surface charge σ.



monopole:

$$q = \text{total charge} = a^2 \sigma$$

dipole:

$$\vec{p} = \int d^3r \, \rho(\vec{r}) \, \vec{r} \qquad \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \sigma \int_{-\frac{a}{2}}^{a/2} dx \int_{-\frac{a}{2}}^{a/2} dy \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

evaluated at $z=0$

as charge distribution is symmetric about the origin

quadrupole

$$\overleftrightarrow{Q}_{ij} = \int d^3r \left[3 r_i r_j - r^2 \delta_{ij} \right] \rho(r)$$

$$= \sigma \int_{-\frac{a}{2}}^{a/2} dx \int_{-\frac{a}{2}}^{a/2} dy \left[3 r_i r_j - r^2 \delta_{ij} \right]$$

evaluated at $z=0$

$r_1 = x, \; r_2 = y, \; r_3 = z$

Because all the charge is at $z=0$ the only components of $\overleftrightarrow{Q}$ that are not zero are

$$\overleftrightarrow{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix}$$

ie $Q_{xz} = 0$

$Q_{yz} = 0$

$Q_{zx} = 0$

$Q_{zy} = 0$

since, for example,

$$Q_{xz} = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \left[3x \cdot \overset{\text{since } z=0}{0} - (x^2+y^2+\overset{\text{since } z=0}{0^2})\, \underbrace{\delta_{xz}}_{=0 \text{ since } \delta_{ij}=0 \text{ for } i \neq j} \right]$$

$$= 0$$

But $Q_{zz} \neq 0$!

$$Q_{zz} = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \left[3 \cdot 0 \cdot 0 - (x^2+y^2)\, \underbrace{\delta_{zz}}_{=1} \right]$$

$$= -\sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \; (x^2+y^2)$$

$$= -\sigma a \left[\int_{-\frac{a}{2}}^{a/2} dx\, x^2 + \int_{-\frac{a}{2}}^{\frac{a}{2}} dy\, y^2 \right]$$

$$= -2a\sigma \left(\frac{x^3}{3} \right)_{-\frac{a}{2}}^{a/2} = -\frac{2a\sigma}{3} \left(\frac{a^3}{8} - \frac{-a^3}{8} \right) = -\frac{4}{3 \cdot 8} a^4 \sigma$$

$$\boxed{Q_{zz} = -\frac{a^4}{6} \sigma}$$

$Q_{xx} = Q_{yy}$ by symmetry

$$= \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \left(3x^2 - (x^2+y^2)\, \underbrace{\delta_{xx}}_{=1} \right)$$

$$= \sigma \int_{-\frac{a}{2}}^{a/2} dx \int_{-a/2}^{a/2} dy \left(2x^2 - y^2 \right)$$

$$= \sigma a \left[2\int_{-\frac{a}{2}}^{\frac{a}{2}} dx\, x^2 - \int_{-\frac{a}{2}}^{a/2} dy\, y^2 \right]$$

$$= \sigma a \left[\frac{x^3}{3} \right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{\sigma a}{3} \left[\frac{a^3}{8} - \frac{-a^3}{8} \right] = \frac{\sigma a^4}{12}$$

Finally

$$Q_{xy} = Q_{yx} = \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \left[x\, y - (x^2+y^2) \underset{0}{\underset{\shortparallel}{\delta_{xy}}} \right]$$

$$= \sigma \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy\; x\, y = 0$$

$$\overleftrightarrow{Q} = \frac{\sigma a^4}{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$V(\hat{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0 r^3} \frac{1}{2} \hat{r} \cdot \overleftrightarrow{Q} \cdot \hat{r}$$

$$\hat{r} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

$$\overleftrightarrow{Q} \cdot \hat{r} = \frac{\sigma a^4}{12} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ -2\cos\theta \end{pmatrix}$$

$$\hat{r}\cdot\overleftrightarrow{Q}\cdot\hat{r} = \frac{\sigma a^4}{12}\left(\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi - 2\cos^2\theta\right)$$

$$= \frac{\sigma a^4}{12}\left(\sin^2\theta - 2\cos^2\theta\right)$$

$\sigma a^2 = q$

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0 r^3}\frac{q a^2}{24}\left(\sin^2\theta - 2\cos^2\theta\right)}$$

Note: at this level of approximation, $V(\vec{r})$ is rotationally invariant about $\hat{z}$ axis (ie $V(\vec{r})$ does not depend on φ) even though the charge distribution is NOT rotationally invariant.

To see the φ dependence of $V(\vec{r})$ we would have to go to higher multipoles.

Note: $\sin^2\theta = 1 - \cos^2\theta$ $\qquad$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

so $\sin^2\theta - 2\cos^2\theta = 1 - 3\cos^2\theta$

$$= 1 - 3\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= -\frac{1}{2} - \frac{3}{2}\cos 2\theta$$

$$= -\frac{1}{2}\left(1 + 3\cos 2\theta\right)$$