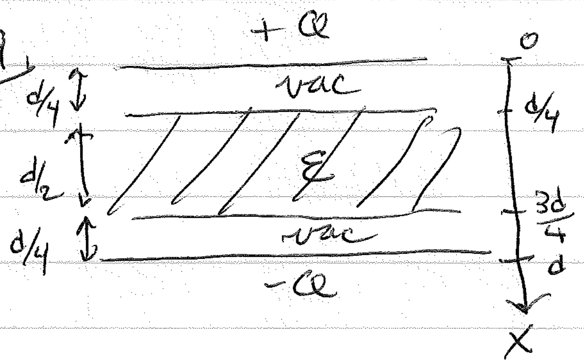
4.19



dielectric fills half of space between capacitor plates as shown

what is the capacitance?

Solve for $\vec{D}$: $\vec{D}(x) = \begin{cases} 0 & x<0,\ x>d \\ \sigma\hat{x} & 0<x<d \end{cases}$

$$\vec{E} = \vec{D}/\varepsilon = \begin{cases} 0 & x<0,\ x>d \\ \frac{\sigma}{\varepsilon_0}\hat{x} & 0<x<d/4 \text{ and } \frac{3d}{4}<x<d \\ \frac{\sigma}{\varepsilon}\hat{x} & \frac{d}{4}<x<\frac{3d}{4} \end{cases}$$

voltage drop $\Delta V = V_+ - V_- = -\int_d^0 \vec{E}\cdot d\vec{l} = \int_0^d \vec{E}\cdot d\vec{l} = \int_0^d dx\, E(x)$

$$= \left(\frac{\sigma}{\varepsilon_0}\right)\frac{d}{4} + \left(\frac{\sigma}{\varepsilon}\right)\frac{d}{2} + \left(\frac{\sigma}{\varepsilon_0}\right)\frac{d}{4}$$

$$= \frac{\sigma d}{2}\left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right]$$

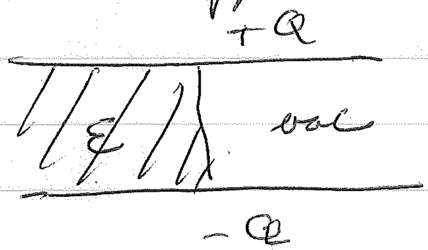
capacitance

$$C_a = \frac{Q}{\Delta V} = \frac{\sigma A}{\frac{\sigma d}{2}\left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right]} = \frac{2A}{d}\,\frac{\varepsilon\varepsilon_0}{\varepsilon + \varepsilon_0}$$

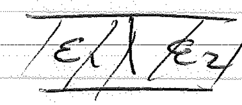
with vacuum between the plate, $C_0 = \frac{A\varepsilon_0}{d}$

$$\text{So } \frac{C_a}{C_0} = \frac{2\varepsilon}{\varepsilon + \varepsilon_0} = \frac{2}{1 + \varepsilon_0/\varepsilon} = \frac{2}{1 + 1/K}$$

Now suppose



capacitor half filled with dielectric as shown

In lecture we found for the case 

$$C = \frac{A}{d}\frac{\varepsilon_1}{2} + \frac{A}{d}\frac{\varepsilon_2}{2}$$

(assuming each dielectric filled half the space)

So apply with $\varepsilon_1 = \varepsilon_0$ for the vacuum
$\varepsilon_2 = \varepsilon$ for the dielectric

$$C_b = \frac{A}{d}\frac{\varepsilon_0}{2} + \frac{A\,\varepsilon}{d\,2} = \frac{A}{2d}(\varepsilon_0 + \varepsilon)$$

$$\frac{C_b}{C_0} = \frac{\frac{A}{2d}(\varepsilon_0+\varepsilon)}{\frac{A}{d}\varepsilon_0} = \frac{1}{2}\left(1 + \frac{\varepsilon}{\varepsilon_0}\right) = \frac{1}{2}(1+K)$$

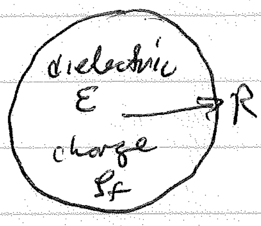
which is bigger?

Consider $$\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+K}{2} - \frac{2K}{1+K} = \frac{(1+K)^2 - 4K}{2(1+K)}$$

$$= \frac{1+K^2+2K-4K}{2(1+K)} = \frac{(1-K)^2}{2(1+K)} > 0 \text{ for any } K$$

Hence the 2nd configuration always increases the capacitance more than the first.

4.20



sphere of dielectric with uniform charge density ρ_f throughout the sphere

find the potential at the center of the sphere

Solve for $\vec{D}$ $\qquad \oint_S \vec{D}(\vec{r})\cdot d\vec{a} = Q_{encl}^{free}$

by symmetry $\vec{D}(\vec{r}) = D(r)\hat{r}$

for Gaussian surface a sphere of radius r

$$\oint_S \vec{D}(r)\cdot d\vec{a} = 4\pi r^2 D(r) = Q_{encl}^{free}$$

$$Q_{encl}^{free} = 4\pi\int_0^r dr' r'^2 \rho_f \qquad \text{for } r < R$$

$$= \frac{4\pi}{3} r^3 \rho_f \qquad \text{for } r < R$$

$$= \frac{4\pi}{3} R^3 \rho_f \qquad \text{for } r > R$$

$$\vec{D}(\vec{r}) = \frac{Q_{encl}^{free}}{4\pi r^2}\hat{r} = \begin{cases} \frac{r}{3}\rho_f\,\hat{r} & r < R \\ \frac{R^3}{3r^2}\rho_f\,\hat{r} & r > R \end{cases}$$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{\vec{D}}{\varepsilon} = \begin{cases} \frac{r}{3\varepsilon}\rho_f\,\hat{r} & r < R \\ \frac{R^3}{3\varepsilon_0 r^2}\rho_f\,\hat{r} & r > R \end{cases}$$

since $\varepsilon = \varepsilon_0$ in vacuum

$$V(0) = -\int_{\infty}^{0} \vec{E}(\vec{r})\cdot d\vec{l} = \int_{0}^{\infty} \vec{E}(\vec{r})\cdot d\vec{l}$$ take outward radial path

$$= \int_0^R dr\, \frac{r}{3\varepsilon}\rho_f + \int_R^\infty dr\, \frac{R^3}{3\varepsilon_0 r^2}\rho_f$$

$$= \frac{R^2}{6\varepsilon}\rho_f + \frac{R^3\rho_f}{3\varepsilon_0 R} = \left(\frac{1}{6\varepsilon} + \frac{1}{3\varepsilon_0}\right) R^2\rho_f$$

$$V(0) = \frac{R^2\rho_f}{3\varepsilon_0}\left(1 + \frac{1}{2}\frac{\varepsilon_0}{\varepsilon}\right) = \frac{R^2\rho_f}{3\varepsilon_0}\left(1 + \frac{1}{2K}\right)$$

where is the bound charge? see next page

where is the bound charge?

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\varepsilon_0 \chi_e r}{3\varepsilon} \rho_f \hat{r} \quad \text{inside sphere}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\frac{\varepsilon_0 \chi_e}{3\varepsilon}\rho_f \vec{\nabla}\cdot(r\hat{r})$$

$$= -\frac{\varepsilon_0 \chi_e \rho_f}{3\varepsilon} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot r\right)$$

$$= -\frac{\varepsilon_0 \chi_e \rho_f}{\varepsilon} = -\frac{(\varepsilon - \varepsilon_0)}{\varepsilon}\rho_f = -\left(1 - \frac{1}{\kappa}\right)\rho_f$$

using $\varepsilon_0 \chi_e = \varepsilon - \varepsilon_0$

use $\varepsilon = \varepsilon_0(1+\chi_e) = \varepsilon_0 + \varepsilon_0 \chi_e \Rightarrow \varepsilon_0 \chi_e = \varepsilon - \varepsilon_0$

$$\kappa = 1 + \chi_e \quad \text{so} \quad 1 - \frac{1}{\kappa} = \frac{\kappa - 1}{\kappa} = \frac{\chi_e}{1+\chi_e}$$

$$\rho_b = -\frac{\chi_e}{1+\chi_e}\rho_f$$ which we found earlier is true in general.

But there is also a surface bound charge at R

$$\sigma_b = \hat{n}\cdot\vec{P} = \hat{r}\cdot\frac{\varepsilon_0 \chi_e R}{3\varepsilon}\rho_f \hat{r} = \frac{\varepsilon_0 \chi_e}{3\varepsilon}\rho_f R$$

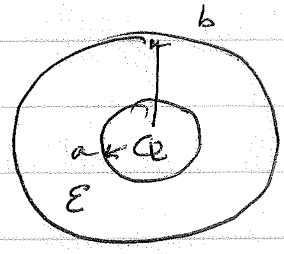
$$\sigma_b = \frac{\rho_f}{3}\left(\frac{\chi_e}{1+\chi_e}\right)R$$

Can check that total bound charge vanishes

$$\frac{4}{3}\pi R^3 \rho_b + 4\pi R^2 \sigma_b = -\frac{4}{3}\pi R^3\left[\frac{\chi_e}{1+\chi_e}\rho_f\right] + 4\pi R^2\left[\frac{\rho_f}{3}\left(\frac{\chi_e}{1+\chi_e}\right)R\right]$$

$$= 0$$

4.26 →



charge Q is uniformly distributed on surface of sphere radius a. What is work done to create this configuration?

$\vec{D}(r) = D(r)\hat{r}$

$$\vec{D}(\vec{r}) = \begin{cases} \dfrac{Q}{4\pi r^2}\hat{r} & a < r \\ 0 & r < a \end{cases}$$

$$\vec{E} = \vec{D}/\varepsilon = \begin{cases} 0 & r < a \\ \dfrac{Q\,\hat{r}}{4\pi\varepsilon r^2} & a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2} & b < r \end{cases}$$

$$W = \frac{1}{2}\int d^3r\, \vec{D}\cdot\vec{E} = \frac{4\pi}{2}\int_a^b dr\, r^2 \left(\frac{Q}{4\pi r^2}\right)^2 \frac{1}{\varepsilon}$$

$$+ \frac{4\pi}{2}\int_b^\infty dr\, r^2 \left(\frac{Q}{4\pi r^2}\right) \frac{1}{\varepsilon_0}$$

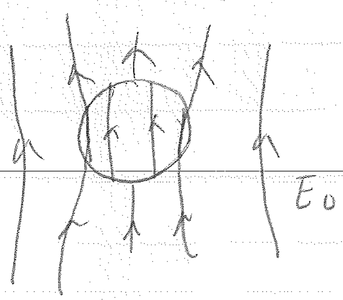
$$= \frac{2\pi}{\varepsilon_0}\left(\frac{Q}{4\pi}\right)^2 \left\{ \frac{\varepsilon_0}{\varepsilon}\int_a^b dr\, \frac{1}{r^2} + \int_b^\infty dr\, \frac{1}{r^2} \right\}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left\{ \frac{\varepsilon_0}{\varepsilon}\left(-\frac{1}{r}\right)_a^b + \left(-\frac{1}{r}\right)_b^\infty \right\}$$

$$W = \frac{Q^2}{8\pi\varepsilon_0}\left\{ \frac{\varepsilon_0}{\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b} \right\}$$

Ex 7.1

dielectric sphere in uniform $\vec{E}_0 = E_0 \hat{z}$



we know that if $\vec{P}$ is uniform in sphere, then $\vec{E}$ will be uniform in sphere. Since applied field is uniform, this is the likely situation

$\Rightarrow$ suppose total field in sphere is $\underset{\text{in}}{\vec{E}} = \underset{\substack{\uparrow \\ \text{applied}}}{\vec{E}_0} + \underset{\substack{\uparrow \\ \text{due to polarization} \\ \text{of sphere}}}{\vec{E}_1}$

sphere is polarized with $\vec{P} = \varepsilon_0 \chi_e \vec{E}_{in} = \varepsilon_0 \chi_e (\vec{E}_0 + \vec{E}_1)$

$\vec{P}$ produces field $\vec{E}_1 = \dfrac{-\vec{P}}{3\varepsilon_0}$

$$\Rightarrow -\frac{\varepsilon_0 \chi_e (\vec{E}_0 + \vec{E}_1)}{3\varepsilon_0} = \vec{E}_1$$

Solve for $\vec{E}_1$: $\quad \vec{E}_0 + \vec{E}_1 = -\dfrac{3}{\chi_e}\vec{E}_1 \Rightarrow \vec{E}_0 = -\left(1 + \dfrac{3}{\chi_e}\right)\vec{E}_1$

$$\Rightarrow \vec{E}_1 = \frac{-\vec{E}_0}{1 + 3/\chi_e}$$

total field <u>inside</u> is $\vec{E}_0 + \vec{E}_1 = \vec{E}_0 \left(1 - \dfrac{1}{1+3/\chi_e}\right) = \vec{E}_0 \left(\dfrac{3/\chi_e}{1+3/\chi_e}\right)$

$$= \vec{E}_0 \left(\frac{1}{1 + \chi_e/3}\right)$$

Outside, total field is $\vec{E} = \vec{E}_0 + \dfrac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

$\uparrow$ dipole field in spherical coords: sphere at origin

total dipole moment: $\quad \vec{p} = \frac{4}{3}\pi R^3 \vec{P}$

$$= \tfrac{4}{3}\pi R^3 (-3\varepsilon_0 \vec{E}_1)$$

$$= \frac{+4\pi R^3 \varepsilon_0 \vec{E}_0}{1 + 3/\chi_e} \Rightarrow \vec{E} = \vec{E}_0 + \frac{R^3 E_0}{(1+3/\chi_e) r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

inside $\vec{E} = \vec{E}_0 \left(\frac{1}{1+\chi_e/3} \right)$

outside $\vec{E} = \vec{E}_0 + \frac{R^3 E_0}{(1+3/\chi_e)\, r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$

inside

$\vec{D} = \varepsilon \vec{E} = \vec{E}_0 \left(\frac{\varepsilon}{1+\chi_e/3} \right)$

outside $\vec{D} = \varepsilon_0 \vec{E} = \varepsilon_0 \vec{E}_0 + \frac{\varepsilon_0 R^3}{(1+3/\chi_e)\, r^3} (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$

check that $\vec{D}\cdot\hat{n}$ is continuous $(\hat{n} \equiv \hat{r})$.

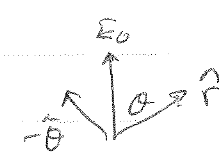
inside $\vec{D}\cdot\hat{r} = \frac{\varepsilon}{1+\chi_e/3} \vec{E}_0 \cdot \hat{r} = \frac{\varepsilon E_0 \cos\theta}{1+\chi_e/3} = \boxed{\frac{\varepsilon_0 (1+\chi_e) E_0 \cos\theta}{1+\frac{\chi_e}{3}} = (\vec{D}\cdot\hat{r})_{inside}}$

outside $\vec{D}\cdot\hat{r} = \varepsilon_0 \vec{E}_0 \cdot \hat{r} + \frac{\varepsilon_0 E_0}{1+3/\chi_e} 2\cos\theta$

$= \varepsilon_0 E_0 \cos\theta \left(1 + \frac{2}{1+3/\chi_e} \right) = \varepsilon_0 E_0 \cos\theta \left(\frac{3+3/\chi_e}{1+3/\chi_e} \right)$

$= \boxed{\varepsilon_0 E_0 \cos\theta \left(\frac{\chi_e + 1}{\frac{\chi_e}{3} + 1} \right) = (\vec{D}\cdot\hat{r})_{outside}}$

so $\vec{D}\cdot\hat{n}$ is continuous



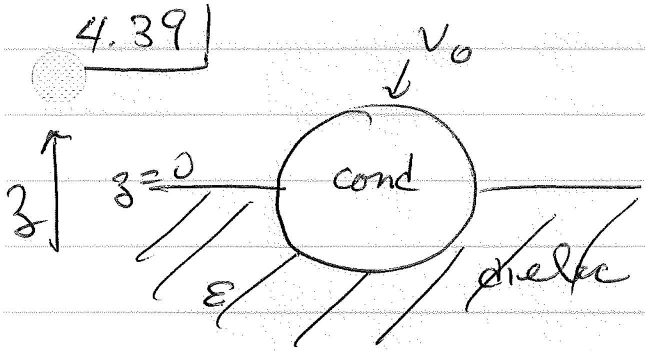
Check that tangential $\vec{E}$ is continuous

inside: $\vec{E}\cdot\hat{\theta} = \frac{\vec{E}_0 \cdot \hat{\theta}}{1+\chi_e/3} = \boxed{\frac{-\sin\theta E_0}{1+\chi_e/3} = (\vec{E}\cdot\hat{\theta})_{in}}$

outside: $\vec{E}\cdot\hat{\theta} = \vec{E}_0 \cdot \hat{\theta} + \frac{E_0}{1+3/\chi_e} \sin\theta = E_0 \sin\theta \left(-1 + \frac{1}{1+3/\chi_e} \right)$

$= E_0 \sin\theta \left(\frac{-1 - 3/\chi_e + 1}{1+3/\chi_e} \right) = \boxed{-E_0 \sin\theta \left(\frac{1}{\frac{\chi_e}{3}+1} \right) = (\vec{E}\cdot\hat{\theta})_{out}}$

4.39



conducting sphere at fixed potential V_0 embedded in dielectric as shown

a) As the problem states, lets assume the potential $V(\vec{r})$ is the same as it would be if there was no dielectric

Then, $\boxed{V(\vec{r}) = \frac{V_0 R}{r}}$ (since $V(r) \sim \frac{1}{r}$ and $V(R) = V_0$)

We can then find the resulting electric field

$$\boxed{\vec{E}(\vec{r}) = \frac{V_0 R}{r^2}\hat{r}} \quad r > R, \quad \vec{E} = 0 \text{ inside sphere}$$

the polarization in the dielectric $z<0$ would then be

$$\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \chi_e \frac{V_0 R}{r^2}\hat{r}}$$

The bound charge in the dielectric is then

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\varepsilon_0 \chi_e V_0 R \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = 0$$

$$\sigma_b = \hat{n}\cdot\vec{P} = -\hat{r}\cdot\vec{P}(R) = -\varepsilon_0 \chi_e \frac{V_0 R}{R^2} = -\varepsilon_0 \chi_e \frac{V_0}{R}$$

$$\boxed{\rho_b = 0, \quad \sigma_b = -\varepsilon_0 \chi_e \frac{V_0}{R}}$$

To get the free charge we use

$$(\vec{D}_{above} - \vec{D}_{below}) \cdot \hat{n} = \sigma_f$$

Because the surface of the conductor is held at potential V_0 there is necessarily free charge on the surface of the conductor. That is the only free charge in the problem.

$\vec{D}_{above}$ is just outside surface of conductor
$\vec{D}_{below}$ is just inside surface of conductor
$\vec{D}_{below} = 0$ since $\vec{E} = 0$ in conductor

For $z > 0$, $\vec{D} = \varepsilon_0 \vec{E}$

$$\varepsilon_0 \vec{E}_{above} \cdot \hat{r} = \frac{\varepsilon_0 V_0 R}{R^2} = \boxed{\frac{\varepsilon_0 V_0}{R} = \sigma_f \, (z>0)}$$

For $z < 0$, $\vec{D} = \varepsilon \vec{E}$

$$\varepsilon \vec{E}_{above} \cdot \hat{r} = \boxed{\frac{\varepsilon V_0}{R} = \sigma_f \, (z<0)}$$

so σ_{free} is not uniformly distributed over the surface of the conductor. BUT the total surface charge is!

$$z > 0, \quad \sigma_{tot} = \sigma_f(z>0) = \frac{\varepsilon_0 V_0}{R}$$

$z<0, \ \sigma_{tot} = \sigma_f(z<0) + \sigma_b$

$$= \frac{\varepsilon V_0}{R} - \frac{\varepsilon_0 \chi_e V_0}{R}$$

recall $\varepsilon = \varepsilon_0(1+\chi_e) = \varepsilon_0 + \varepsilon_0\chi_e$ so $\varepsilon_0\chi_e = \varepsilon - \varepsilon_0$

$z<0 \quad \sigma_{tot} = \frac{\varepsilon V_0}{R} - \frac{(\varepsilon-\varepsilon_0)V_0}{R} = \boxed{\frac{\varepsilon_0 V_0}{R} = \sigma_{tot}}$

so σ_{tot} is the same both for $z\geq 0$ and $z\leq 0$!

b) The only charge in the problem is the uniform σ_{tot} on the surface of the conducting sphere

let $Q = 4\pi R^2 \sigma_{tot} = 4\pi R^2 \frac{\varepsilon_0 V_0}{R} = 4\pi R \varepsilon_0 V_0$

then $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{4\pi R\varepsilon_0 V_0}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{R V_0}{r^2}\hat{r}$

we get the same $\vec{E}$ as found in part (a)

We can also check that all other boundary conditions are satisfied

on surface of conductor: jump in D gave us σ_f
$E \perp$ surface $\Rightarrow$ tangential components of $\vec{E}$ vanish & so are continuous across surface

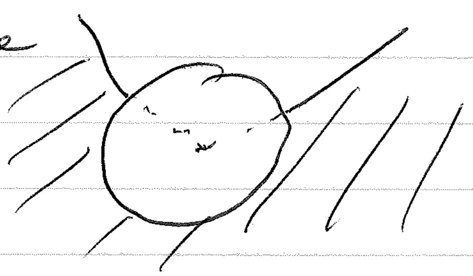
at interface $z=0$

$\vec{E}$ and $\vec{D}$ are tangential to the surface so normal component of $\vec{D} = 0$ on both sides $\Rightarrow$ consistent with $\sigma_f = 0$ on this surface.

tangential component of $\vec{E}$ continuous as cross surface at $z=0$.

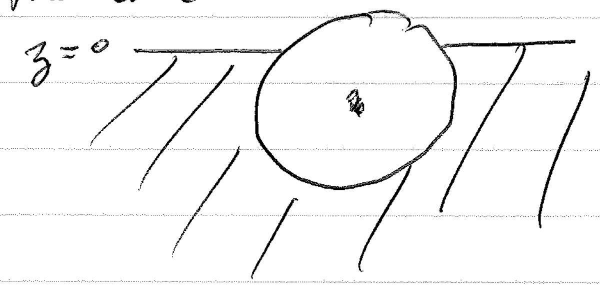
$\Rightarrow$ We found a solution that is consistent with all the charges and boundary conditions so it is the solution

For the case



solution will work the same way

For the case



it does not work since $\vec{D}\cdot\hat{z}$ would be discontinuous at $z=0$ interface, but there is no free charge there