If the wire is cut, setting current $I = 0$, what is the net charge that flows around the loop?

Initially, there is a magnetic flux $\Phi$ through the loop. After the wire is cut, the flux drops to $\Phi_f = 0$. The changing flux creates an induced emf and current in the loop that drives current in the loop that causes charge to flow. We have

$$\text{induced emf} \quad \mathcal{E} = -\frac{d\Phi}{dt}$$

Since $\Phi$ is decreasing, $\frac{d\Phi}{dt} < 0 \implies \mathcal{E} > 0$.

$\Rightarrow$ current flows counter-clockwise around loop.

Current $\Rightarrow \quad I = \frac{\mathcal{E}}{R} = -\frac{1}{R} \frac{d\Phi}{dt}$

Total charge passing a given point in the loop is

$$Q = \int_0^t I dt = \int_0^t \frac{\mathcal{E}}{R} dt = \int_0^t \frac{1}{R} \frac{d\Phi}{dt} dt$$

$$= -\frac{1}{R} [\Phi_f - \Phi_i] = \frac{\Phi_i}{R}$$

$$Q = \frac{\Phi_i}{R}$$
It remains to find $E_c$.

Flux through loop $S$ due to magnetic field from wire:

\[
\mathcal{F}_B = \oint_S \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \quad \text{by symmetry, } \mathbf{B}(r) = B(r) \hat{r}
\]

\[
\mathcal{F}_B = 2\pi R B(R) = \mu_0 I
\]

Cylindrical radial coord.

\[
B(r) = \frac{\mu_0 I}{2\pi r}
\]

Flux through loop in $E_c = \oint_S \mathbf{a}_n \cdot \mathbf{B} \quad \text{at a point in } S$

\[
E_c = \frac{\mu_0 I}{2\pi} \left[ \ln \left( \frac{R}{s} \right) \right]
\]

\[
= \frac{\mu_0 s \ln 2 I}{2\pi}
\]

So

\[
\mathcal{F} = \frac{\mu_0 s \ln 2 I}{2\pi R}
\]
Betaatron: use $\frac{d^2r}{dt^2}$ to accelerate a charge in a circular orbit.

Magnetic field $\vec{B} = B(r) \hat{z}$

$B$ is cylindrically symmetric about $\hat{z}$.

\[ \text{cyclotron} \text{ cyclotron motion of charged particle in circular orbit at radius } r. \]

\[ m\frac{d^2r}{dt^2} = q \frac{\vec{v} \times \vec{E}}{r} = -q v B(r) \hat{r} \]

(For $q > 0$, $\vec{v}$ must be in $-\hat{r}$ direction, so charge moves clockwise, in order for Lorentz force to be in $-\hat{r}$ direction)

$\Rightarrow m\vec{v} = qr B(r)$ determines velocity $\vec{v}$ of charge for given orbit.

Now suppose $B(r)$ changes with time $t$ → magnetic flux through orbit of charge changes $\Rightarrow$ electric field $\vec{E}$ is induced that accelerates the charge.

What condition must hold for the charge to be accelerated, but stay in fixed orbit at radius $r$?

Let us find the induced $\vec{E}$. For $\vec{B} = B(r) \hat{z}$

symmetry gives $\vec{E} = E(r) \hat{r}$ where $E$ depends only on the cylindrical radial distance $r$, and points in radial direction $\hat{r}$.

[Hint: $\vec{B} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\frac{\partial \vec{B}}{\partial t} \sim$ like current flowing down wire, $\vec{E}$ is like resulting magnetic field, $\vec{D} = \rho \vec{J}$]
For \( \vec{E}(r) = \vec{E}(r) \hat{\phi} \)

\[
\vec{V} \times \vec{E} = \frac{1}{r} \frac{d}{dr} (r \vec{E}(r)) \hat{\phi}
\]

in cylindrical coordinates

\[
= - \frac{\partial B}{\partial t} \hat{\phi}
\]

\[
\Rightarrow \frac{d}{dr} (r \vec{E}(r)) = - r \frac{\partial B}{\partial t}
\]

\[
\pi \int_0^r dr' \frac{d}{dr} (r' \vec{E}(r')) = - 2\pi \int dr' r' \frac{\partial B}{\partial t} = - \frac{\partial \Phi}{\partial t}
\]

where \( \Phi \) is the flux through the orbit of radius \( r \).

Define the average magnetic field over the orbit, \( B_{av} \), by

\[
\Phi = \pi r^2 B_{av}
\]

Integrating the left hand side, we get

\[
2\pi r \vec{E}(r) = - \pi r^2 \frac{dB_{av}}{dt}
\]

\[
\vec{E}(r) = - \frac{r}{2} \frac{dB_{av}}{dt}
\]

\( \vec{E} \) in the direction for which it is going around the orbit. We have for

the magnitude of \( \vec{E} \)

\[
\frac{m \vec{v}}{dt} = q \vec{E} = \frac{q}{2} \frac{dB_{av}}{dt}
\]

charge moves \( \vec{E} \) points \( \frac{q}{2} \frac{dB_{av}}{dt} \) \( \Phi \) direction

clockwise \( \frac{q}{2} \frac{dB_{av}}{dt} \) \( \Phi \) direction

- \( \vec{E} \) direction

- \( \frac{q}{2} \frac{dB_{av}}{dt} \) direction
\[ m \frac{\text{d}v}{\text{d}t} = q e \frac{\text{d}B_{av}}{\text{d}t} \]

But to maintain circular orbit the cyclotron condition is

\[ m v = q e B(r) \Rightarrow m \frac{\text{d}v}{\text{d}t} = q e \frac{\text{d}B(r)}{\text{d}t} \text{ if } r \text{ stays constant} \]

\[ \Rightarrow \frac{1}{2} \frac{\text{d}B_{av}}{\text{d}t} = \frac{\text{d}B(r)}{\text{d}t} \]

The magnetic field \( B(r) \) at the radius of the orbit should be \( \frac{1}{2} \) the average magnetic field averaged over the area of the orbit.

\[ B(r) = \frac{1}{2} B_{av} \]
Self inductance of a solenoid length $l$, radius $R$

\[ \mathbf{B} = \mu_0 N I \hat{z} \]

times area per unit length.

Total flux through all the wire loops that make up the solenoid is

\[ \Phi = (\mu_0 N I \pi R^2) (N l) = \mu_0 N^2 \pi R^2 I \]

flux through one loop

\[ \Phi = l I \] defines self inductance $L$

\[ L = \mu_0 N^2 \pi R^2 \]

Another way to do the calculation:

The energy stored in this magnetostatic configuration is

\[ W_{mag} = \frac{1}{2} \mu_0 \int d^3r \, |\mathbf{B}|^2 = \frac{1}{2} \mu_0 (\mu_0 N I)^2 (\pi R^2 l) \]

we assume only $B$ we need

consider is that inside B-field in volume, inside solenoid

\[ W_{mag} = \frac{1}{2} \mu_0 N^2 \pi R^2 I^2 \]

But we also have \[ W_{mag} = \frac{1}{2} l I^2 \] \[ L = \mu_0 N^2 \pi R^2 \]

some result as above.
Let's find the capacitance and inductance of the plates.

a) Capacitance: Suppose charge $+Q$ is distributed uniformly over top plate and $-Q$ over bottom plate. There is voltage drop $V$ between plates.

Capacitance is $C = \frac{Q}{V}$

Surface charge on plates are $+\sigma$ and $-\sigma$ where $\sigma = Q/\ell w$

Resultant electric field $E = \frac{\sigma}{\varepsilon_0} \hat{z}$

Voltage $V = E S = \sigma S = \frac{Q S}{\ell w \varepsilon_0}$

Capacitance is $C = \frac{Q}{V} = \frac{\ell w \varepsilon_0}{\sigma}$

Capacitance per unit length is $C = \frac{C}{S} = \frac{\varepsilon_0}{\ell w}$

b) Inductance: Suppose we have uniform sheet current $I = K x$ on top plate and $-K x$ on bottom plate.
\[ B = \mu_0 \nabla \times H \]

\[ \vec{B} = \mu_0 \nabla \times \vec{H} = \mu_0 \frac{I}{w} \hat{y} \]

Magnetic field between plates is uniform.

\[ \Phi = B l s = \mu_0 l s \frac{I}{w} \]

\[ = LI \Rightarrow \begin{aligned} L &= \frac{\mu_0 l s}{w} \quad \text{self-inductance} \end{aligned} \]

Inductance per length, \( L = \frac{L}{s} = \frac{s}{w} \mu_0 \)

a) Now suppose the plates are charged up with \( Q \), and then connected by a wire into a closed circuit to allow the plates to discharge.

The resulting circuit is just like an RC circuit.

\[ L \begin{aligned} \frac{Q}{C} \end{aligned} \]

In problem 4.24, done in workshop, you found that the current (and hence charge on the capacitor) in such a circuit oscillates with a frequency

\[ \omega = \frac{1}{\sqrt{LC}} \quad \text{for our case} \]

\[ \omega = \frac{1}{\sqrt{(\frac{\mu_0 l s}{w})(\frac{k_0}{s} \varepsilon_0)}} = \frac{1}{\sqrt{k_0^2 \mu_0 \varepsilon_0}} = \frac{1}{2 \sqrt{k_0 \varepsilon_0}} \]
Experimentally it is found that \( \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c \) the speed of light in a vacuum.

Hence \( \omega = \frac{c}{L} \) or \( L = \frac{c}{\omega} \)

Such a geometry is called a transmission line and an electric pulse is found to travel down the transmission line with velocity \( v = \frac{1}{\sqrt{\varepsilon_0 c}} \).
Maxwell's Equations

\[ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Amperes law for magnetostatics was

\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]

but \[ \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} = -\mu_0 \frac{\partial \vec{B}}{\partial t} \]

by charge conservation.

By general theorem, \[ \nabla \cdot \vec{B} = 0 \]

unless have electostatics + magnetostatics.

\[ \Rightarrow \text{Amperes law can't be valid outside static situations} \]

To fix: write \[ -\mu_0 \frac{\partial \vec{B}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = \nabla \cdot (\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \]

corrected.

So \[ \nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot (\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = 0 \]

\[ \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \vec{B} \text{ displacement current} \]

\[ \oint \vec{B} \cdot d\vec{a} = \mu_0 I_{\text{end}} + \mu_0 \varepsilon_0 \int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} \]

so \[ \frac{\partial \vec{E}}{\partial t} \text{ is a source of } \vec{B} \text{, just like } \frac{\partial \vec{B}}{\partial t} \text{ is a source of } \vec{E} \]
\[ \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} \]

\[ \oint \mathbf{d} \mathbf{l} \cdot \mathbf{B} = \mu_0 I_{\text{end}} + \mu_0 \varepsilon_0 \int \mathbf{dA} \cdot \frac{\partial \mathbf{E}}{\partial t} \]

"find \( \mathbf{B} \) between plates for \( \mathbf{Ia} \)"

charge on plates \( Q = \mathbf{I} t \) on left plate, \( -\mathbf{I} t \) on right plate

\( \Rightarrow \mathbf{E} \) between plates \( \hat{\mathbf{E}} = \frac{\mathbf{E}}{\varepsilon_0} \hat{x} = \frac{\varepsilon_0}{\pi a^2 \varepsilon_0} \hat{x} \)

\[ \mathbf{E} = \frac{\mathbf{I} t}{\varepsilon_0 \pi a^2} \hat{x} \rightarrow \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{I}}{\varepsilon_0 \pi a^2} \hat{x} \]

in region between plates, symmetry \( \Rightarrow \hat{\mathbf{B}} = B(r) \hat{\phi} \)

Take loop of radius \( r \) centered about wire, between plates

\[ \oint \mathbf{d} \mathbf{l} \cdot \mathbf{B} = 2 \pi r B(r) = \mu_0 I_{\text{end}} + \mu_0 \varepsilon_0 \int \mathbf{dA} \cdot \frac{\mathbf{I}}{\varepsilon_0 \pi a^2} \hat{x} \]

area of loop = \( \pi r^2 \)

\[ = \mu_0 \varepsilon_0 \frac{\pi r^2 \mathbf{I}}{\varepsilon_0 \pi a^2} \]

\[ \mathbf{B}(r) = \frac{\mu_0 \mathbf{I}}{2 \pi a^2} \frac{r \hat{\phi}}{r} \]

just like we had a wire with uniform current density

\[ \mathbf{J} = \frac{\mathbf{I}}{\pi a^2} \hat{x} \]
For \( r > a \), if ignore "edge" effects from
non uniformity of \( \mathbf{E} \) at edges of plates

\[
\oint \mathbf{B} \cdot d\mathbf{s} = 2\pi r B(r) = \mu_0 I \text{ and } + \mu_0 \int_0^\infty \frac{I}{\pi a^2} \frac{x}{x^2} e^{-2x} dx
\]

\[
= \mu_0 I
\]

\[\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi}\]
just like around
wire with current \( I \)

do as above, took area of loop as

but could also take any area bounded by curve,

\[\text{now } \oint \mathbf{dA} \cdot \mathbf{E} = 0 \text{ on this area}
\]

\[\text{but } \mathbf{I}_{\text{end}} = I\]

so

\[\mathbf{B} = \frac{\mu_0 I \hat{\phi}}{2\pi r} \text{ as before}\]