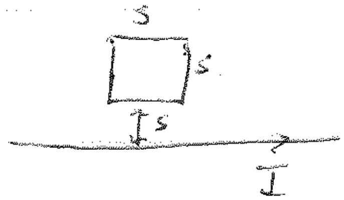
7.18



square loop, sides s, has resistance R
distance from current carrying wire is s.

If the wire is cut, sending current $I \to 0$, what is the net charge that flows around the loop?

Initially, there is a magnetic flux Φ_i through the loop. After the wire is cut, this flux drops to $\Phi_f = 0$. This changing flux creates an induced emf around the loop that drives current in the loop that causes charge to flow. We have

induced emf $\quad \mathcal{E} = -\dfrac{d\Phi}{dt}$

Since Φ is decreasing, $\dfrac{d\Phi}{dt} < 0 \Rightarrow \mathcal{E} > 0$
$\Rightarrow$ current flows counterclockwise around loop

current is $\quad I = \dfrac{\mathcal{E}}{R} = -\dfrac{1}{R}\dfrac{d\Phi}{dt}$

total charge passing a given point in the loop is

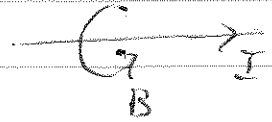
$$Q = \int_0^\infty I\,dt = \int_0^\infty \frac{\mathcal{E}}{R}\,dt = -\int_0^\infty \frac{1}{R}\frac{d\Phi}{dt}\,dt$$

$$= -\frac{1}{R}\left[\Phi_f - \Phi_i\right] = \frac{\Phi_i}{R}$$

$$\boxed{Q = \frac{\Phi_i}{R}}$$

It remains to find Φ_i

Flux through loop is due to magnetic field from wire



by symmetry, $\vec{B}(\vec{r}) = B(r)\hat{\varphi}$

cylindrical radial coord.

$$\oint \vec{B} \cdot d\hat{l} = 2\pi r B(r) = \mu_0 I$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi r} I \hat{\varphi}$$

Flux through loop is $\Phi_i = \int_{\square} d\vec{a} \cdot B$ $\quad$ $d\vec{a}$ points in $\hat{\varphi}$ direction

$$\Phi_i = s\int_s^{2s} dr\, B(r) = s\int_s^{2s} dr\, \frac{\mu_0}{2\pi r} I = \frac{\mu_0 s I}{2\pi}\left[\ln 2s - \ln s\right]$$

$$= \frac{\mu_0 s}{2\pi} \ln 2 \; I$$

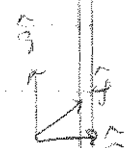
So $\boxed{Q = \frac{\mu_0 s \ln 2}{2\pi R} I}$

7.50 Betatron: use $\frac{\partial B}{\partial t}$ to accelerate a charge in a circular orbit.

magnetic field $\vec{B} = B(r)\hat{z}$ r is cylindrical radial coord

B is cylindrically symmetric about $\hat{z}$

cylindrical coords



~~cyclotron~~ cyclotron motion of charged particle in circular orbit at radius r.

$$m\vec{a} = -\frac{mv^2}{r}\hat{r} = q\vec{v}\times\vec{B} = -qvB(r)\hat{r}$$

(For $q>0$, $\vec{v}$ must be in $-\hat{\varphi}$ direction, ie charge moves clockwise, in order for Lorentz force to be in $-\hat{r}$ direction)

$\Rightarrow mv = qrB(r)$ determines velocity v of charge for circular orbit.

Now suppose $B(r)$ changes with time $\rightarrow$ magnetic flux through orbit of charge changes $\Rightarrow$ electric field $\vec{E}$ is induced that accelerates the charge.

What condition must hold for the charge to be accelerated, but stay in fixed orbit at radius r?

Let us find the induced $\vec{E}$. For $\vec{B} = B(r)\hat{z}$ symmetry gives $\vec{E} = E(r)\hat{\varphi}$ $\vec{E}$ depends only on the cylindrical radial distance r, and points in polar direction $\hat{\varphi}$. [Hint: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\frac{\partial\vec{B}}{\partial t}$ is like current flowing down wire, $\vec{E}$ is like resulting magnetic field $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$]

For $\vec{E}(\vec{r}) = E(r)\hat{\varphi}$

$$\vec{\nabla}\times\vec{E} = \frac{1}{r}\frac{d}{dr}(rE(r))\ \hat{z} \quad \text{in cylindrical coords}$$

$$= -\frac{\partial B}{\partial t}\ \hat{z}$$

$$\Rightarrow \quad \frac{d}{dr}\Big(rE(r)\Big) = -r\frac{\partial B}{\partial t}$$

$$2\pi\int_0^r dr'\frac{d}{dr'}(r'E(r')) = -2\pi\int_0^r dr'\, r'\frac{\partial B}{\partial t} = -\frac{\partial \Phi}{\partial t}$$

where Φ is the flux through the orbit of radius r. Define the average magnet field over this orbit, B_{av}, by $\Phi = \pi r^2 B_{av}$.

Integrating left hand side we get

$$2\pi r E(r) = -\pi r^2\frac{dB_{av}}{dt}$$

$$E(r) = -\frac{r}{2}\frac{dB_{av}}{dt} \qquad \text{for } \frac{dB_{av}}{dt} > 0,\ \vec{E} \propto -\hat{\varphi} \text{ direction}$$

This $\vec{E}$ will accelerate the charge's velocity with which it is going around the orbit. We have for the magnitude of $\vec{v}$

$$m\frac{dv}{dt} = qE = +q\frac{r}{2}\frac{dB_{av}}{dt}$$

charge moves clockwise in $-\hat{\varphi}$ direction

$\vec{E}$ points clockwise in $-\hat{\varphi}$ direction

$q\frac{r}{2}\frac{dB_{av}}{dt}$ gives magnitude of E

$$m \frac{dv}{dt} = q \frac{r}{2} \frac{dB_{av}}{dt}$$

But to maintain circular orbit the cyclotron condition is

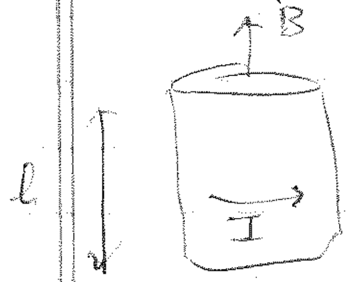
$$mv = qr\,B(r) \Rightarrow m\frac{dv}{dt} = qr\frac{\partial B(r)}{\partial t} \quad \text{if } r \text{ stays constant}$$

$$\Rightarrow \frac{1}{2}\frac{dB_{av}}{dt} = \frac{\partial B(r)}{\partial t}$$

ie the magnetic field $B(r)$ at the radius of the orbit should be $\frac{1}{2}$ the ~~average~~ magnetic field averaged over the area of the orbit

$$\boxed{B(r) = \frac{1}{2} B_{av}}$$

Self inductance of a solenoid length l, radii R



$$\vec{B} = \mu_0 N I \hat{z}$$

$\uparrow$ # turns of wire per unit length

Total flux through all the wire loops that make up the solenoid is

$$\Phi = \underbrace{(\mu_0 N I \pi R^2)}_{\text{flux through one loop}} \underbrace{(N l)}_{\text{number of loops}} = \mu_0 N^2 l \pi R^2 I$$

$\Phi = L I$ defines self inductance L

$$\Rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$$

Another way to do the calculation:

The energy stored in this magnetostatic configuration is

$$W_{mag} = \frac{1}{2\mu_0} \int d^3r \, |\vec{B}|^2 = \frac{1}{2\mu_0} \underbrace{(\mu_0 N I)^2}_{\text{B-field in solenoid}} \underbrace{(\pi R^2 l)}_{\text{volume inside solenoid}}$$

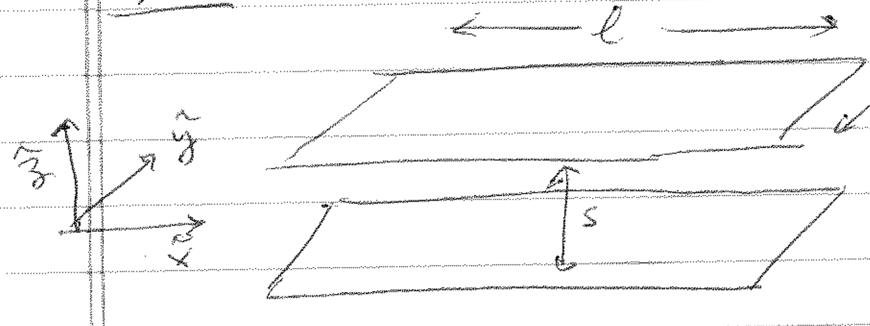
(we assume only B we need consider is that inside the solenoid)

$$W_{mag} = \frac{1}{2} \mu_0 N^2 l \pi R^2 \, I^2$$

But we also know $W_{mag} = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$

same result as above

7.62



parallel plates length l, width w separated distance s

Lets find the capacitance and inductance of the plates

a) Capacitance Suppose charge $+Q$ is distributed uniformly over top plate and $-Q$ over bottom plate. There is voltage drop V between plate.

Capacitance is $C = \dfrac{Q}{V}$

Surface charge on plates are $+\sigma$ and $-\sigma$ where $\sigma = Q/lw$

Resulting electric field is $\vec{E} = \dfrac{\sigma}{\varepsilon_0}(-\hat{z})$

voltage is $V = Es = \dfrac{\sigma s}{\varepsilon_0} = \dfrac{Qs}{lw\varepsilon_0}$

Capacitance is

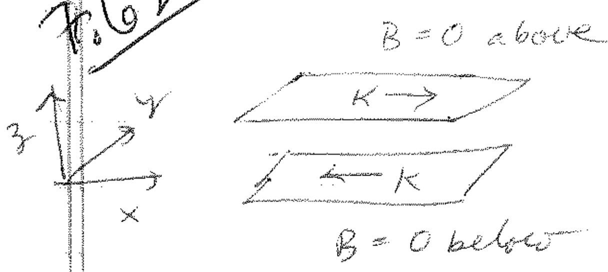
$$C = \frac{Q}{V} = \frac{lw\varepsilon_0}{s}$$

Capacitance per unit length is

$$\mathcal{C} = \frac{C}{l} = \frac{w\varepsilon_0}{s}$$

b) Inductance: Suppose we have uniform sheet current $\vec{K} = K\hat{x}$ on top plate and $-K\hat{x}$ on bottom plate

P.62



magnetic field between plates is uniform

$$\vec{B} = \mu_0 K \hat{y} = \mu_0 \frac{I}{w} \hat{y}$$

$Kw = I$ total current

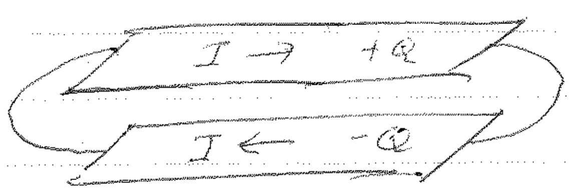
magnetic flux between the plates is

$$\Phi = Bls = \frac{\mu_0 l s}{w} I$$

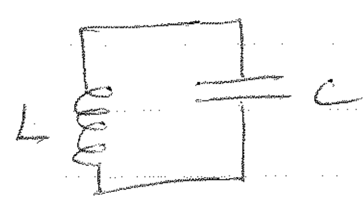
$$= LI \Rightarrow \boxed{L = \frac{\mu_0 l s}{w} \quad \text{self inductance}}$$

inductance per length $\mathcal{L} = \frac{L}{l} = \frac{s}{w}\mu_0$

c) Now suppose the plates are charged up with Q, and then connected by a wire into a closed circuit to allow the plates to discharge



The resulting circuit is just like an LC circuit



For prob 7.27, done in workshop, you found that the current (and hence charge on the capacitor) in such a circuit oscillates with a frequency

$$\omega = \frac{1}{\sqrt{LC}} \quad \text{for our case}$$

$$\omega = \frac{1}{\sqrt{\left(\frac{\mu_0 l s}{w}\right)\left(\frac{l w}{s}\varepsilon_0\right)}} = \frac{1}{\sqrt{l^2 \mu_0 \varepsilon_0}} = \frac{1}{l\sqrt{\mu_0 \varepsilon_0}}$$

experimentally it is found that $\frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c$ the speed of light in a vacuum!

Hence $\omega = \frac{c}{l}$ or $l = \frac{c}{\omega}$

Such a geometry is called a transmission line and an electric pulse is found to travel down the transmission line with velocity $v = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} = c$

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Ampere's law for magnetostatics was

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

but $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \mu_0\vec{\nabla}\cdot\vec{J} = \underbrace{-\mu_0\frac{\partial\rho}{\partial t}}_{\neq 0}$ by charge conservation

($=0$ by general theorem of vector calculus; $\neq 0$ unless have electrostatics + magnetostatics)

$\Rightarrow$ Ampere's law can't be valid outside static situations

To fix: write $-\mu_0\frac{\partial\rho}{\partial t} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = \vec{\nabla}\cdot\left(-\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$

correction is

So $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot\left(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$

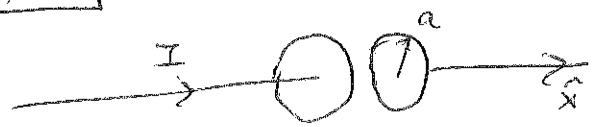
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \underbrace{\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{"displacement current"}}$$

$$\oint_P \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\varepsilon_0\int_S\left(\frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{a}$$

so $\frac{\partial\vec{E}}{\partial t}$ is a source of $\vec{B}$, just like $\frac{\partial\vec{B}}{\partial t}$ is a source of $\vec{E}$

7.35

7.31



$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int d\vec{a} \cdot \frac{\partial \vec{E}}{\partial t}$$

find B between plates for $r<a$.

charge on plates $Q = It$ on left plate, $-It$ on right plate

$\Rightarrow$ E between plates is $\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{x} = \frac{Q}{\pi a^2 \varepsilon_0} \hat{x}$

$$\vec{E} = \frac{It}{\varepsilon_0 \pi a^2} \hat{x} \rightarrow \frac{\partial \vec{E}}{\partial t} = \frac{I}{\varepsilon_0 \pi a^2} \hat{x}$$

in region between plates, symmetry $\Rightarrow \vec{B} = B(r)\hat{\phi}$

Take loop of radius r centered about wire, inbetween plates

$$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r) = \mu_0 \underset{0}{\underset{\shortparallel}{I_{encl}}} + \mu_0 \varepsilon_0 \int d\vec{a} \cdot \frac{I}{\varepsilon_0 \pi a^2} \hat{x}$$

area of loop $= \pi r^2$

$$= \mu_0 \varepsilon_0 \frac{\pi r^2 I}{\varepsilon_0 \pi a^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 r}{2\pi a^2} I \hat{\phi}$$

just like we had a wire with uniform current density $\vec{J} = \frac{I}{\pi a^2} \hat{x}$

for $r > a$, if ignore "edge" effects from non uniformity of $\vec{E}$ at edges of plates

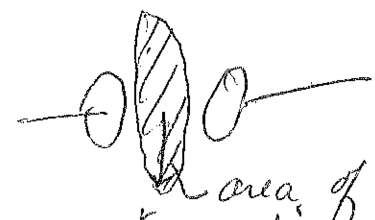
$$\oint d\vec{\ell}\cdot\vec{B} = 2\pi r\, B(r) = \mu_0 \underbrace{I_{encl}}_{0} + \cancel{\mu_0\varepsilon_0 \int d\vec{a}\cdot\frac{I}{\varepsilon_0 \pi a^2}\hat{x}}$$

$$+\ \mu_0\varepsilon_0(\pi a^2)\frac{I}{\varepsilon_0 \pi a^2}$$

since only area between plates has $\frac{\partial E}{\partial t} \neq 0$

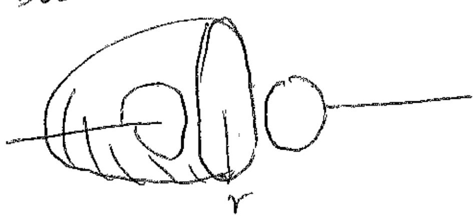
$$= \mu_0 I$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$ just like around wire with current I

to do above, took area of loop as 

area of integration

but could also take any area bounded by curve,



now $\int d\vec{a}\cdot\frac{\partial \vec{E}}{\partial t} = 0$ on this area

but $I_{encl} = I$

So $B = \frac{\mu_0 I \hat{\phi}}{2\pi r}$ as before