Energy + Momentum Conservation (7.5)

We say in electrostatics $W_{elec} = \frac{\varepsilon_0}{2}\int d^3r\; E^2$

magnetostatics $W_{mag} = \frac{1}{2\mu_0}\int d^3r\; B^2$

Now treat for full electrodynamic situation

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ $\qquad$ $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \frac{\partial\vec{E}}{\partial t}\mu_0\varepsilon_0$ $\qquad$ $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\mathcal{E}$ emf

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voltage drop

power dissipated in current carrying wire is VI ← total current

$V = EL$ $\qquad$ $I = Aj$

↑ electric field in wire $\qquad$ ↑ cross sectional area

$\Rightarrow$ power ~~dissipated~~ is $EjLA = Ej\,\text{vol}$

in general power dissipated $= \frac{d}{dt}$ (mechanical or chemical work done on system)

$$\frac{dW_{mech}}{dt} = \int_{vol} d^3r\; \vec{E}\cdot\vec{j}$$

also can get this: work done to move charge $d\vec{r}$ is

$$W = \vec{F}\cdot d\vec{r} = (q\vec{E} + q\vec{v}\times\vec{B})\cdot d\vec{r}$$

work per unit time is

$$\frac{dW}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v} = q\vec{E}\cdot\vec{v} + \underbrace{q(\vec{v}\times\vec{B})\cdot\vec{v}}_{=0}$$

work per unit time done on all charges is

$$\frac{dW}{dt} = \int d^3r\; \underbrace{n(\vec{r})}_{\text{density of charges}}\, q\,\vec{v}\cdot\vec{E} = \int d^3r\; \vec{j}\cdot\vec{E}$$

Mechanical energy $\equiv$ kinetic energy of moving charges

$$W = \frac{1}{2} m v^2$$

$$\frac{dW}{dt} = m \vec{v} \cdot \frac{d\vec{v}}{dt} \qquad \text{Newton} \quad m \frac{d\vec{v}}{dt} = \vec{F}$$

$$= \vec{v} \cdot \vec{F} = \vec{v} \cdot \left(q\vec{E} + q\, \vec{v} \times \vec{B} \right)$$

$$= q\, \vec{v} \cdot \vec{E}$$

for many charges

$$\frac{dW}{dt} = \int d^3r \; \underbrace{q\, n(\vec{r})\, \vec{v}(r)}_{\vec{j}(\vec{r})} \cdot \vec{E}(r)$$

$$= \int d^3r \; \vec{j} \cdot \vec{E}$$

$$\frac{dW_m}{dt} = \int d^3r\, \vec{J}\cdot\vec{E}$$

Ampere's Law $\vec{J} = \dfrac{\vec{\nabla}\times\vec{B}}{\mu_0} - \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$\frac{dW_m}{dt} = \int d^3r \left[\underbrace{\frac{\vec{E}\cdot(\vec{\nabla}\times\vec{B})}{\mu_0}}_{\substack{\text{integrate} \\ \text{by parts}}} - \underbrace{\varepsilon_0 E \cdot \frac{\partial \vec{E}}{\partial t}}_{= \frac{1}{2}\frac{\partial E^2}{\partial t}} \right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\frac{dW_m}{dt} = \int d^3r \left[\frac{1}{\mu_0}\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \varepsilon_0 \frac{1}{2}\frac{\partial E^2}{\partial t} \right]$$

use $\vec{\nabla}\times\vec{E} = \dfrac{-\partial \vec{B}}{\partial t}$ Faraday

use $\vec{B}\cdot\dfrac{\partial \vec{B}}{\partial t} = \dfrac{1}{2}\dfrac{\partial B^2}{\partial t}$

$$= \int d^3r \left[\left(\frac{-1}{2}\right)\left(\frac{\partial B^2}{\mu_0 \partial t} + \varepsilon_0 \frac{\partial E^2}{\partial t}\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

$$\frac{dW_m}{dt} = -\frac{d}{dt}\int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2 \right) - \frac{1}{\mu_0}\oint_{Surface} d\vec{a}\cdot(\vec{E}\times\vec{B})$$

define $W_{EB} = \displaystyle\int_{Vol} d^3r \left(\frac{1}{2\mu_0}B^2 + \frac{\varepsilon_0}{2}E^2 \right)$ electro-magneto energy in volume V

$\vec{S} \equiv \dfrac{\vec{E}\times\vec{B}}{\mu_0}$ "Poynting vector"

= energy density current

$$\frac{dW_m}{dt} = -\frac{d W_{EB}}{dt} - \oint d\vec{a} \cdot \vec{S}$$

increase in mechanical energy (kinetic energy of charges) = energy lost from $\vec{E} + \vec{B}$ fields − energy from $E + B$ fields flowing out of volume through surface

write $u_{EB} = \frac{\varepsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$ energy density of electromagnetic fields

u_M = mechanical energy density

$$\frac{d}{dt}\int_V d^3r\, u_m + \frac{d}{dt}\int_V d^3r\, u_{EB} = -\oint_S \vec{S}\cdot d\vec{a} = -\int_V d^3r\, \vec{\nabla}\cdot\vec{S}$$

$$\frac{\partial}{\partial t}\left(u_M + u_{EB}\right) = -\vec{\nabla}\cdot\vec{S}$$

law of local conservation of energy for e-m fields

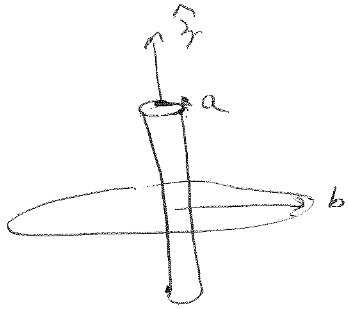
(Same form as charge conservation $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}$

$\vec{S}$ is flux of energy carried by $\vec{E} + \vec{B}$ fields

$\oint_S \vec{S}\cdot d\vec{a}$ is energy per unit time carried by $\vec{E} + \vec{B}$ fields through surface S

8-13

~~7-6~~



a) Field in solenoid is $\vec{B} = \mu_0 N I_s \hat{z}$, $\vec{B}=0$ outside from solenoid

Flux through ring is $\Phi = \oint \vec{B}\cdot d\vec{a} = B\pi a^2 = \mu_0 \pi a^2 N I_s$

emf around ring (going in $\hat{\phi}$ direction)

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 \pi a^2 N \frac{dI_s}{dt}$$

current in ring is $\vec{I}_r = \frac{\mathcal{E}}{R}\hat{\phi} = -\frac{\mu_0 \pi a^2 N}{R}\frac{dI_s}{dt}\hat{\phi}$

b) power delivered to ring is $I_r^2 R = \left(\mu_0 \pi a^2 N \frac{dI_s}{dt}\right)^2 \frac{1}{R} \equiv P$

must come from solenoid.

Show that power ~~and~~ carried away from solenoid by $\vec{E} + \vec{B}$ fields is just P above.

energy flux is $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$. Calculate $\vec{S}$ on outside surface of solenoid.

<u>integral form of Faraday's law</u>

$\vec{E}$ is produced by the $\frac{\partial \vec{B}}{\partial t}$. Since $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int d\vec{a}\cdot\vec{B}$

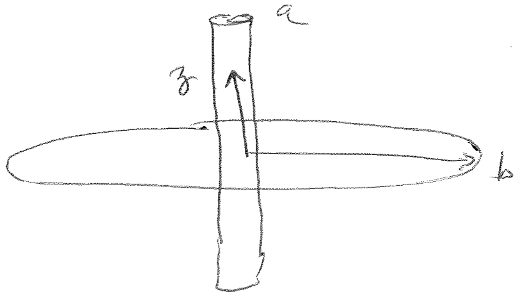
evaluate on path of radius $r=a$ just outside solenoid

By symmetry, $\vec{E} = E(r)\hat{\phi}$

$$\oint \vec{E}\cdot d\vec{l} = 2\pi a E(a) = -\frac{d}{dt}\Phi = -\mu_0 \pi a^2 N \frac{dI_s}{dt}$$

$$\vec{E} = \frac{-\mu_0 \pi a^2 N}{2\pi a}\frac{dI_s}{dt}\hat{\phi}$$

the $\vec{B}$ just outside the solenoid, is the $\vec{B}$ produced by I_r flowing in the ring (I_s in solenoid produces no $\vec{B}$ outside $r \geq a$)



since $b \gg a$, we can approx $\vec{B}$ just outside solenoid, at height z, by the value at the center of the ring + height z. See example 6 Chpt 5 for result:

$$\vec{B}(z) = \frac{\mu_0 I_r}{2} \frac{b^2 \hat{z}}{(b^2+z^2)^{3/2}} = \frac{\mu_0}{2}\left(-\frac{\mu_0 \pi a^2 N}{R}\frac{dI_s}{dt}\right)\frac{b^2 \hat{z}}{(b^2+z^2)^{3/2}}$$

$\Rightarrow$ Poynting vector $\quad \vec{S} = \frac{1}{\mu_0}\left(\frac{-\mu_0 \pi a^2 N}{2\pi a}\frac{dI_s}{dt}\right)\frac{\mu_0}{2}\left(\frac{-\mu_0 \pi a^2 N}{R}\frac{dI_s}{dt}\right)$

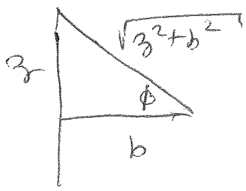
$$\times \frac{b^2}{(b^2+z^2)^{3/2}} \underbrace{(\hat{\phi} \times \hat{z})}_{=\hat{r}}$$

$$\vec{S} = \left(\mu_0 \pi a^2 N \frac{dI_s}{dt}\right)^2 \frac{1}{2\pi a R} \frac{b^2 \hat{r}}{2(b^2+z^2)^{3/2}}$$

total power leaving solenoid is just $\oint_S \vec{S} \cdot d\vec{a}$ where S is outside surface of solenoid

$$\oint \vec{S} \cdot d\vec{a} = 2\pi a \int_{-\infty}^{\infty} dz\, \vec{S}(z) \cdot \hat{r}$$

$$= \left(\mu_0 \pi a^2 N \frac{dI_s}{dt}\right)^2 \frac{1}{R} \frac{b^2}{2} \int_{-\infty}^{\infty} dz \frac{1}{(b^2+z^2)^{3/2}}$$

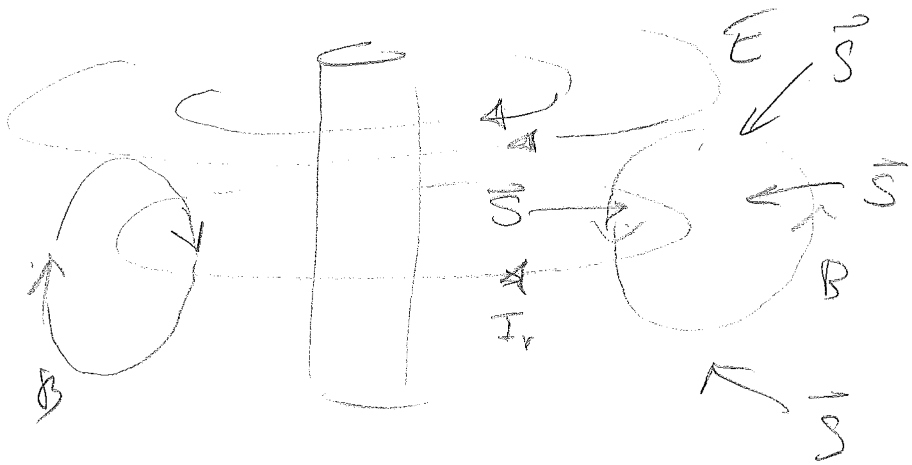


$b\tan\phi = z \Rightarrow dz = \frac{b}{\cos^2\phi}\,d\phi$

$$\frac{1}{\sqrt{z^2+b^2}} = \frac{\cos\phi}{b}$$

$$\int_{-\infty}^{\infty} dz\,\frac{1}{(b^2+z^2)^{3/2}} = \int_{-\frac{\pi}{2}}^{\pi/2} d\phi\,\frac{b}{\cos^2\phi}\,\frac{\cos^3\phi}{b^3} = \int_{-\frac{\pi}{2}}^{\pi/2} d\phi\,\frac{\cos\phi}{b^2} = \frac{2}{b^2}$$

$\Rightarrow \oint \vec{S}\cdot d\vec{a} = \left(\mu_0 \pi a^2 N \frac{dI_s}{dt}\right)^2 \frac{1}{R}$ = P power absorbed by ring.



$$\vec{E} = \frac{\mathcal{E}}{2\pi a}\,\hat{\phi} \qquad \vec{B} = \frac{\mu_0}{2} I_r \frac{b^2\,\hat{z}}{(b^2+z^2)^{3/2}}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\mathcal{E} I_r}{2\pi a}\,\frac{1}{2}\,\frac{b^2}{(b^2+z^2)^{3/2}}\,\hat{r}$$

$$\oint \vec{S}\cdot d\vec{a} = \mathcal{E} I_r \int_{-\infty}^{\infty} dz\,\frac{b^2}{2\,(b^2+z^2)^{3/2}} = \mathcal{E} I_r$$