Maxwell's Eqn in potential form

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$ remains true with dynamics

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\vec{\nabla}\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V$$

$$\boxed{\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}}$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0} \quad (*)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \Rightarrow \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}} = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}} \quad (**)$$

gauge transformations: if $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ then $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}$

$\vec{A}'$ gives same $\vec{B}$ as $\vec{A}$.

But if we change $\vec{A}\to\vec{A}'$, we also have to change $V\to V'$ so $\vec{E}$ stays same.

~~If $V' = V + a$, const a then $-\vec{\nabla}V' = -\vec{\nabla}V$~~

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V - \frac{\partial \vec{A}'}{\partial t} + \vec{\nabla}\left(\frac{\partial\lambda}{\partial t}\right)$$

$$= -\vec{\nabla}\left(V - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial\vec{A}'}{\partial t}.$$ If let $V' = V - \frac{\partial\lambda}{\partial t}$ then

$$\vec{E} = -\vec{\nabla}V' - \frac{\partial\vec{A}'}{\partial t}$$ has same form as before

$\Rightarrow$ the transformation $\left.\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\lambda \\ V' &= V - \frac{\partial \lambda}{\partial t} \end{aligned}\right\}$ called a "gauge" transformation

for any scalar function $\lambda(\vec{r}, t)$, leaves $\vec{B}$ and $\vec{E}$ unchanged.
ie:

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$$
$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t}$$

We can therefore use this freedom, given by the arbitrary λ, to make $\vec{\nabla} \cdot \vec{A}$ equal to some desired quantity, which will simplify the equations $(*)(**)$ Making such a choice for $\vec{\nabla} \cdot \vec{A}$ is called "fixing" the gauge.

i) <u>Coulomb gauge</u>: same as used in magnetostatics.

Choose $\vec{\nabla} \cdot \vec{A} = 0$

(if had some $\vec{A}'$ such that $\vec{\nabla} \times \vec{A}' = \vec{B}$, but $\vec{\nabla} \cdot \vec{A}' \neq 0$, ~~then~~ we could always find a $\lambda(\vec{r}, t)$ such that $\vec{A} = \vec{A}' + \vec{\nabla}\lambda$ does satisfy $\vec{\nabla} \cdot \vec{A} = 0$) see Griffiths sec 5.4.1

$(*) \Rightarrow \nabla^2 V = -\rho/\varepsilon_0$

solution is $V(\vec{r}, t) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int d^3r' \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$

} same as in electrostatics

but unlike statics, need also to know $\vec{A}$ in order to get $\vec{E}$.

$$(**) \Rightarrow \left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) = -\mu_0 \vec{J} + \mu_0 \varepsilon_0 \vec{\nabla}\left(\frac{\partial V}{\partial t}\right)$$

$$= -\mu_0 \vec{J} + \frac{\mu_0 \varepsilon_0}{4\pi \varepsilon_0} \vec{\nabla} \int d^3 r' \left(+\frac{\partial \rho}{\partial t}(r',t)\right) \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= -\mu_0 \vec{J} + \frac{\mu_0 \varepsilon_0}{4\pi \varepsilon_0} \vec{\nabla} \int d^3 r' \frac{\left[-\vec{\nabla}' \cdot \vec{J}(r',t)\right]}{|\vec{r}-\vec{r}'|}$$

$\vec{A}$ and $\vec{J}$ solve an "intergral - differential" equation.

2) <u>Lorentz gauge</u>: Choose $\vec{\nabla} \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$

(can always find $\lambda(r,t)$ such that this will be true)see prob 10.6 for proof in Lorentz gauge: $(*) \Rightarrow \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0$

$$(**) \Rightarrow \nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

equations for V and $\vec{A}$ have the same form

d'Alambertian operator $\Box^2 \equiv \nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}$ wave equation operator

$$\begin{cases} \Box^2 V = -\rho/\varepsilon_0 \\ \Box^2 \vec{A} = -\mu_0 \vec{J} \end{cases}$$

henceforth we will use Lorentz gauge for non-static problems

<u>Lorentz force</u>

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) = q\Big(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} + \underbrace{\vec{v}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}}\Big)$$

$$= -q\Big(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} - \vec{\nabla}(\vec{v}\cdot\vec{A})\Big)$$

$$= -q\Big(\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} + \vec{\nabla}(V - \vec{v}\cdot\vec{A})\Big)$$

$\frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A} = \frac{d A}{dt}$ convective derivative

$$\frac{d}{dt}\left(\vec{A}(\vec{r}(t),t)\right) = \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{A}}{\partial y}\frac{dy}{\partial t} + \frac{\partial\vec{A}}{\partial z}\frac{dz}{\partial t}$$

$$= \frac{\partial\vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$

change in $\vec{A}$ as seen by a particle moving with velocity $\vec{v}$

$$\Rightarrow \frac{d\vec{p}}{dt} = -q\left(\frac{d\vec{A}}{dt} + \vec{\nabla}\left(V - \vec{v}\cdot\vec{A}\right)\right)$$

$$\frac{d}{dt}\Big(\underbrace{\vec{p} + q\vec{A}}_{\text{"canonical" momentum } \vec{p}_{can}}\Big) = -\vec{\nabla}\Big(\underbrace{qV - q\,\vec{v}\cdot\vec{A}}_{\text{"potential" } U}\Big)$$

$$\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}U$$

Note: in Coulomb gauge we had

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t} = -\mu_0 \vec{J} + \frac{\mu_0}{4\pi} \vec{\nabla} \int d^3r' \frac{[-\vec{\nabla}' \cdot \vec{J}(r',t)]}{|\vec{r}-\vec{r}'|}$$

look at right hand side.

see Appendix B Griffiths

From Helmholtz Theorem Corollary, ~~Chpt 1.6.1~~, we saw that an vector function $\vec{F}(\vec{r})$ which $\to 0$ sufficiently rapidly as $r \to \infty$, can be written as:

(B.10) $$\vec{F}(\vec{r}) = \underbrace{\vec{\nabla}\left(\frac{-1}{4\pi}\int d^3r' \frac{\vec{\nabla}' \cdot \vec{F}(r')}{|\vec{r}-\vec{r}'|}\right)}_{\vec{F}_L} + \underbrace{\vec{\nabla} \times \left(\frac{1}{4\pi}\int d^3r' \frac{\vec{\nabla}' \times \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)}_{\vec{F}_T}$$

$\vec{F}_L$ longitudinal part or curlfree part $\vec{\nabla} \times \vec{F}_L = 0$

$\vec{F}_T$ transverse part or divergenceless part $\vec{\nabla} \cdot \vec{F}_T = 0$

Now comparing above, we see

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \mu_0 \vec{J}_L$$ ← longitudinal part of $\vec{J}$

$$\boxed{\square^2 \vec{A} = -\mu_0 \vec{J}_T}$$ ← transverse part of $\vec{J}$

in Coulomb gauge, the source for $\vec{A}$ is the transverse part, or divergenceless part, of the current.