General solution to wave equation

$$\Box^2 f(\vec{r},t) = 0 \qquad \text{subst in F.T.}$$

$$\Box^2 \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \, \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \, \tilde{f}(\vec{k},\omega) \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right) = \vec{\nabla}\cdot\begin{pmatrix} \frac{\partial}{\partial x} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial y} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial z} e^{i(k_x x + k_y y + k_z z)} \end{pmatrix}$$

$$= \vec{\nabla}\cdot\begin{pmatrix} ik_x \\ ik_y \\ ik_z \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right)$$

product rule:

$$= i\vec{k}\cdot\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\Rightarrow \left[\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

So wave eqn is

$$\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \, \tilde{f}(\vec{k},\omega) \left(-k^2 + \frac{\omega^2}{v^2}\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left(-k^2 + \frac{\omega^2}{v^2}\right) = 0$$: if a function $=0$, then its F.T. $=0$ (all Fourier coefficients $=0$)

either

$\Rightarrow$ $\tilde{f}(\vec{k},\omega) = 0$, ie there is no "$\vec{k},\omega$" component in solution

or $k^2 v^2 = \omega^2$. $\therefore$ for each $\vec{k}$, only $\omega = v|\vec{k}|$ is present

So most general solution is

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \quad \tilde{f}(\vec{k}) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{where } \omega^2 = v^2 k^2$$

[ω is not arbitrary + integrated over, only $\omega = v|\vec{k}|$ enters] and $\tilde{f}(\vec{k})$ is anything

General solution is ~~sum~~ linear combination of sinusoidal waves.

In most problems therefore, it will be enough to determine how plane sinusoidal waves with wavevector $\vec{k}$ behave. The general solution can then always be represented as a linear combination of these plain sinusoidal waves.

Alternatively, if $f(r, t=0) = f_0(\vec{r})$ given function, then this will evolve in time according to

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \; \tilde{f}_0(k) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \omega^2 = v^2 k^2$$

$$\text{where} \quad \tilde{f}_0(\vec{k}) = \int \frac{d^3r}{(2\pi)^3} \, f_0(\vec{r}) \, e^{-i\vec{k}\cdot\vec{r}}$$

Inhomogeneous wave equation

$\Box^2 f(\vec{r},t) = g(\vec{r},t)$ where g is a given source function

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$g(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

substitute in

$$\Rightarrow \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\vec{k},\omega)\left[-k^2 + \frac{\omega^2}{v^2}\right] e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \tilde{g}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

equate Fourier components

$$\Rightarrow \tilde{f}(\vec{k},\omega)\left[-k^2 + \frac{\omega^2}{v^2}\right] = \tilde{g}(\vec{k},\omega)$$

$$\Rightarrow \tilde{f}(\vec{k},\omega) = \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}$$

$$\vec{f}(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{\tilde{g}(\vec{k},\omega)}{\frac{\omega^2}{v^2} - k^2}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \; \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{\frac{\omega^2}{v^2} - k^2} \cdot \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3r' \int dt' \; g(\vec{r}\,',t')\, e^{-i(\vec{k}\cdot\vec{r}\,'-\omega t')}$$

$$f(\vec{r},t) = \int d^3k \cancel{\int d\omega \frac{1}{(2\pi)^4} \frac{1}{\frac{\omega^3}{v^2} - k^2}} \int d^3r' e$$

$$= \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \, g(\vec{r}\,', t') \boxed{\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{1}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \, e^{-i\omega(t-t')}}{\left(\frac{\omega^2}{v^2} - k^2\right)}}$$

Green's function for wave equ

$G(\vec{r}-\vec{r}\,', t-t')$

$$f(\vec{r},t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' \; G(\vec{r}-\vec{r}\,', t-t') \, g(\vec{r}\,', t')$$

Polarization

vector wave $\vec{f}(\vec{r},t) = \vec{A}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

where $\vec{A} = A\hat{n}$, A may be complex number to include phase factor.

if $\hat{n}$ is $\parallel \hat{k}$ we have <u>longitudinal</u> polarization

" $\hat{n}$ is $\perp \hat{k}$ " " <u>transverse</u> polarization

~~Plane EM waves in vacuum~~

$\cancel{\vec{\nabla}\cdot\vec{E} = 0} \qquad \cancel{\vec{\nabla}\cdot\vec{B} = 0}$

$\cancel{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}} \qquad \cancel{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$

<u>Circular polarization</u>: sum of orthogonal transverse polarizations, $\pi/2$ out of phase

Consider $\vec{f}(\vec{r},t) = A\hat{n}_1 e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \underbrace{iA\hat{n}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{A\hat{n}_2 e^{i(\vec{k}\cdot\vec{r}-\omega t+\frac{\pi}{2})}}$

where $\hat{n}_1 \perp \hat{n}_2 \perp \vec{k}$

$$\vec{f} = A(\hat{n}_1 + i\hat{n}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

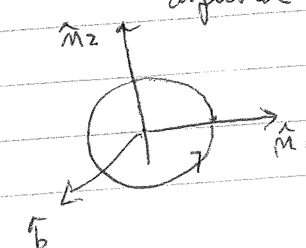
to get physical field, take Real part of complex form

$$\Rightarrow \vec{f} = A\hat{n}_1\cos(\vec{k}\cdot\vec{r}-\omega t) - A\hat{n}_2\sin(\vec{k}\cdot\vec{r}-\omega t)$$

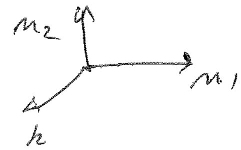
$$= A\hat{n}_1\cos(\omega t - \vec{k}\cdot\vec{r}) + A\hat{n}_2\sin(\omega t - \vec{k}\cdot\vec{r})$$

$$|\vec{f}|^2 = A^2\cos^2(\vec{k}\cdot\vec{r}-\omega t) + A^2\sin^2(\vec{k}\cdot\vec{r}-\omega t) = A^2$$

amplitude constant ~~in time~~; but direction of $\vec{f}$ rotates in time counterclockwise with frequency ω.



$\vec{f} = A(\hat{n}_1 + i\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, where $\hat{n}_1, \hat{n}_2, \hat{k}$ form right handed coord system



is a "Right handed" <u>circularly polarized</u> wave

$\vec{f} = A(\hat{n}_1 - i\hat{n}_2)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is a "Left handed" circularly polarized wave – direction of $\vec{f}$ rotates in time clockwise