Plane EM waves in a vacuum

\[ \nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

Assume solutions of form

\[ \vec{E} = \vec{E}_0 e^{i(k \cdot r - \omega t)} \]
\[ \vec{B} = \vec{B}_0 e^{i(k \cdot r - \omega t)} \]

Maxwell's equations become

1. \( i \vec{k} \cdot \vec{E}_0 = 0 \)
2. \( i \vec{k} \cdot \vec{B}_0 = 0 \)
3. \( i \vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \)
4. \( i \vec{k} \times \vec{B} = \mu_0 \varepsilon_0 (-i \omega) \vec{E}_0 \)

(1) and (3) \( \Rightarrow \) EM waves are transverse polarized. \( \vec{E}_0 \) and \( \vec{B}_0 \) both \( \perp \) to \( \vec{k} \).

2. \( \Rightarrow \vec{B}_0 = \frac{k}{\omega} \times \vec{E}_0 = \frac{1}{c} \vec{k} \times \vec{E}_0 \quad \Rightarrow \vec{B}_0 \perp \vec{E}_0 \)

\[ |\vec{B}_0| = \frac{1}{c} |\vec{E}_0| \]

Very important factor \( \frac{1}{c} \)!

Since Lorentz force is \( \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \), the force on a charged particle due to an electromagnetic wave is predominantly from the electric field \( \vec{E} \). The force due to the magnetic field \( \vec{B} \) \( \approx \vec{v} \times \vec{B}_0 \approx (\frac{v}{c}) \vec{E}_0 \) is reduced by a factor \( (\frac{v}{c}) \ll 1 \) unless charge is moving relativistically fast.
Energy and momentum in EM wave:

\[ \mathbf{E}(r,t) = E_0 \cos(kz - \omega t) \mathbf{\hat{z}} \]

\[ \mathbf{B}(r,t) = \frac{1}{c} E_0 \cos(kz - \omega t) \mathbf{\hat{\ell}} \]

Energy density:

\[ U_{EB} = \frac{E_0^2}{2} \left( \frac{1}{\mu_0} \right) B^2 = \frac{E_0^2}{2} \cos^2(kz - \omega t) + \frac{1}{2\mu_0 \varepsilon_0} E_0^2 \cos^2(kz - \omega t) \]

\[ = \frac{1}{2} E_0^2 \cos^2(kz - \omega t) \left[ \varepsilon_0 + \frac{\mu_0 \varepsilon_0}{\mu_0} \right] \quad \text{use } c^2 = \frac{1}{\mu_0 \varepsilon_0} \]

\[ = \frac{E_0^2}{2 \varepsilon_0} \cos^2(kz - \omega t) \]

Energy current:

\[ \mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \]

\[ = \frac{1}{\mu_0 \varepsilon_0} E_0^2 \cos^2(kz - \omega t) (\mathbf{\hat{z}} \times \mathbf{\hat{\ell}}) = -c E_0^2 \cos^2(kz - \omega t) \mathbf{\hat{\ell}} \]

Using \( \frac{1}{\mu_0 \varepsilon_0} = \frac{c}{\mu_0 \varepsilon_0} = \frac{c \mu_0}{\mu_0} = c \)

\[ \mathbf{S} = c U_{EB} \mathbf{\hat{\ell}} \]

Momentum density:

\[ \mathbf{P}_{EB} = \frac{c}{2} \mathbf{S} = \frac{U_{EB}}{c} \mathbf{\hat{\ell}} \]

\[ \Rightarrow U_{EB} = c |\mathbf{P}_{EB}| \quad \text{energy-momentum relation of photons} \]

Since for visible light \( \lambda \approx 5 \times 10^{-7} \text{ m} \approx 5000 \AA \)

\[ T = \frac{\lambda}{c} = \frac{5 \times 10^{-7}}{3 \times 10^8} \text{ sec} = 1.6 \times 10^{-15} \text{ sec} \]

For most classical measurements, on macroscopic scale,
the measurement will average over many oscillations
of the wave. Therefore one is interested in averages

\[ \langle U_{EB} \rangle = \frac{1}{T} \int_0^T dt \, U_{EB} \]

average over one period of oscillation

\[ = \frac{2}{c} \int_0^T dt \, E_0 E_0^2 \cos^2 (k_x - \omega t) \left[ \frac{T}{2\pi} \right] = \frac{A}{c} \]

\[ \langle U_{EB} \rangle = \frac{1}{2} E_0 E_0^2 \]

average of \( \cos^2(\phi) \) over one period is \( \frac{1}{2} \)

\[ \langle \hat{s} \rangle = c \langle U_{EB} \rangle^\frac{3}{2} \]

\[ \langle \hat{p}_{EB} \rangle = \frac{1}{c} \langle U_{EB} \rangle^\frac{5}{2} \]

\[ \text{intensity} = \text{average power per area transported by wave} \]

\[ I = \langle \hat{s} \rangle \cdot \hat{n} \]

\[ \text{normal to surface through which energy transported} \]

\[ \text{magnitude of energy current} \]

\[ = \text{amplitude of field}^2 \]

\[ \langle \hat{s} \rangle \cdot \hat{n} = \text{average power per area transported through surface with normal} \hat{n} \]
Maxwell's Equations in Matter

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{B} = \mathbf{j}_{\text{tot}} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

want to write \( \mathbf{j}_{\text{tot}} = \mathbf{j}_{\text{free}} + \mathbf{j}_b \)

in statics: \( \mathbf{j}_b = -\nabla \rho \)

\( \mathbf{J}_b = \nabla \times \mathbf{M} \)

in dynamics: conservation of bound charge \( \Rightarrow \nabla \cdot \mathbf{J}_b = -\frac{\partial \rho_b}{\partial t} \)

\[ \nabla \cdot (\nabla \times \mathbf{M}) = \nabla \cdot \nabla \times \mathbf{M} = 0 \]

something must be missing! The bound current arising from \( \mathbf{M} \) must not be all the bound current. There must be bound current arising from a time varying \( \rho \).

bound current from polarization, \( \mathbf{j}_p \) must satisfy

\[ \nabla \cdot \mathbf{j}_p = -\frac{\partial \rho_p}{\partial t} = \frac{2}{\varepsilon_0} \nabla \cdot \rho \]

\[ \Rightarrow \mathbf{j}_p = \frac{\partial \rho}{\partial t} \]

\[ \Rightarrow \mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \rho}{\partial t} \]
\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \left( \rho_f - \nabla \cdot \mathbf{P} \right) \]

\[ \nabla \times \mathbf{B} = \mu_0 \left( \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Define
\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]
\[ \mathbf{H} = \mu_0 \mathbf{B} - \mathbf{M} \]

\[ \Rightarrow \quad \nabla \cdot \mathbf{D} = \rho_f \]
\[ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \]

Homogeneous eqn

\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Homogeneous eqn

For linear materials,
\[ \mathbf{D} = \varepsilon \mathbf{E} \]
\[ \mathbf{H} = \frac{1}{\mu} \mathbf{B} \]

Closes above equations.
If we had \( \vec{D}(r,t) = \varepsilon \vec{E}(r,t) \)
\( \vec{H}(r,t) = \frac{1}{\mu} \vec{B}(r,t) \)

then Maxwell's equations in absence of free charge and free current would be:

\[
\begin{align*}
\varepsilon \nabla \cdot \vec{E} &= 0 \\
\varepsilon \nabla \times \vec{B} &= \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\varepsilon \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}
\end{align*}
\]

everything would be the same except \( \varepsilon \rightarrow \varepsilon \varepsilon_0 \mu_0 \).

The speed of EM waves in the material would be:

\[
\nu = \frac{1}{\sqrt{\varepsilon \mu}} < c \quad \text{c/\nu} \equiv n \quad \text{index of refraction}
\]

would have \( |\vec{B}| = \frac{1}{\nu} |\vec{E}| \)

In general however, things are much more complicated for time varying response.

Consider model for polarization of a neutral atom, that we saw last lecture.

If displace center of electron cloud from ion by distance \( \vec{r} \), then there is a restoring force:

\[
\vec{F}_{\text{rest}} = -\frac{e^2 \vec{r}}{4\pi \varepsilon_0 R^3} = -m \omega_0^2 \vec{r}
\]

\( m \), \( \omega_0 \) has units of mass and frequency.