Plane EM waves in a vacuum

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

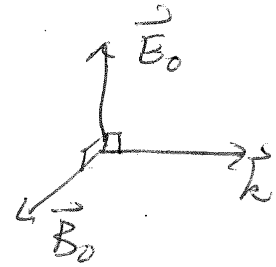
Assume solutions of form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ $\Big\}$ where $\omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}} k = ck$

Maxwell's equ's becom

1) $i\vec{k}\cdot\vec{E}_0 = 0$ 3) $i\vec{k}\cdot\vec{B}_0 = 0$

2) $i\vec{k}\times\vec{E}_0 = +i\omega\vec{B}_0$ 4) $i\vec{k}\times\vec{B} = \mu_0\varepsilon_0(-i\omega)\vec{E}_0$

(1) and (3) $\Rightarrow$ EM waves are transverse polarized. $\vec{E}_0$ and $\vec{B}_0$ both $\perp$ to $\vec{k}$.

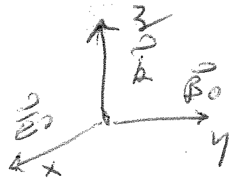


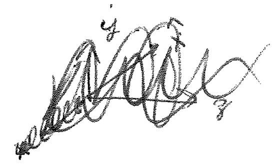
2) $\Rightarrow$ $\vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 = \frac{1}{c}\hat{k}\times\vec{E}_0 \Rightarrow \vec{B}_0 \perp \vec{E}_0$

$|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|$

↑ very important factor $\frac{1}{c}$!

Since Lorentz force is $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, the force on a charged particle due to an electromagnetic wave is predominantly from the electric field $\vec{E}$. The force due to the magnetic field is $\sim vB_0 = \left(\frac{v}{c}\right)E_0$, is reduced by a factor $\left(\frac{v}{c}\right) \ll 1$, unless charge is moving relativistically fast.





<u>energy + momentum</u> in EM wave:

$$\vec{E}(r,t) = E_0 \cos(kz-\omega t)\,\hat{x} \qquad \vec{B}(r,t) = \frac{1}{c}E_0\cos(kz-\omega t)\,\hat{y}$$

energy density

$$u_{EB} = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 = \frac{\varepsilon_0}{2}E_0^2\cos^2(kz-\omega t) + \frac{1}{2\mu_0 c^2}E_0^2\cos^2(kz-\omega t)$$

$$= \frac{1}{2}E_0^2\cos^2(kz-\omega t)\underbrace{\left[\varepsilon_0 + \frac{1}{\mu_0 c^2}\right]}_{\underbrace{\varepsilon_0 + \frac{\mu_0\varepsilon_0}{\mu_0}}_{2\varepsilon_0}} \qquad \text{use } c^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$u_{EB} = \varepsilon_0 E_0^2\cos^2(kz-\omega t)$$

energy current

* Note: when taking the product of 2 factors of $\vec{E}$ or $\vec{B}$, important to take Re parts first, if using complex notation!

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$= \frac{1}{\mu_0 c}E_0^2\cos^2(kz-\omega t)(\hat{x}\times\hat{y}) = c\,\varepsilon_0 E_0^2\cos^2(kz-\omega t)\,\hat{z}$$

using $\frac{1}{\mu_0 c} = \frac{c}{\mu_0 c^2} = \frac{c\,\mu_0\varepsilon_0}{\mu_0} = c\,\varepsilon_0$

$$\vec{S} = c\,u_{EB}\,\hat{z}$$

momentum density $\vec{p}_{EB} = \frac{1}{c^2}\vec{S} = \frac{u_{EB}}{c}\hat{z}$

$\Rightarrow \quad u_{EB} = c\,|\vec{p}_{EB}|$ — energy-momentum relation of photons

Since for visible light, $\lambda \sim 5\times10^{-7}$ m ~ 5000 Å

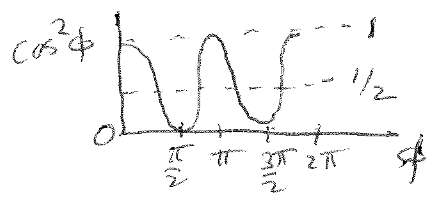
$$T = \frac{\lambda}{c} = \frac{5\times10^{-7}}{3\times10^{8}}\text{ sec} = 1.6\times10^{-15}\text{ sec}$$

for most classical measurements, on macroscopic scale, the measurement will average over many oscillations of the wave. Therefore one is interested in averages

$$\langle u_{EB} \rangle = \frac{1}{T}\int_0^T dt\; u_{EB} \qquad \text{average over one period of oscillation}$$

$$= \frac{\lambda}{c}\int_0^T dt\; \varepsilon_0 E_0^2 \cos^2(kz-\omega t) \qquad \boxed{\begin{aligned} T &= \frac{2\pi}{\omega} \text{ is period of oscillation} \\ &= \frac{\lambda}{c} \end{aligned}}$$

$$\langle u_{EB} \rangle = \frac{1}{2}\varepsilon_0 E_0^2 \qquad \text{average of } \cos^2(\phi) \text{ over one period is } \frac{1}{2}$$



$$\langle \vec{S} \rangle = c\langle u_{EB} \rangle \hat{z}$$

$$\langle \vec{p}_{EB} \rangle = \frac{1}{c}\langle u_{EB} \rangle \hat{z}$$

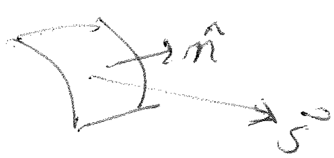
<u>intensity</u> = average (over time) power per area transported by wave

$\cancel{I = \langle \vec{S} \rangle \cdot \hat{n}}$

~~normal to surface through which~~ energy transported

intensity $\quad I = |\langle \vec{S} \rangle|$ $\quad$ magnitude of energy current
~ (amplitude of field)^2

$\langle \vec{S} \rangle \cdot \hat{n}$ = average power per area transported through surface with normal $\hat{n}$



Maxwell's Eqn in Matter

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho_{tot} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_{tot} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

want to write $\rho_{tot} = \rho_{free} + \rho_b \qquad \vec{J}_{tot} = \vec{J}_{free} + \vec{J}_b$

in statics : $\rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \vec{J}_b = \vec{\nabla}\times\vec{M}$

in dynamics : conservation of bound charge $\Rightarrow \vec{\nabla}\cdot\vec{J}_b = -\frac{\partial\rho_b}{\partial t}$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M})}_{=0} = \underbrace{+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})}_{\neq 0}$$

something must be missing! The bound ~~charge~~ current arising from $\vec{M}$ must not be all the bound current. There must be bound current arising from a time varying $\vec{P}$.

bound current from polarization, $\vec{J}_p$ must satisfy

$$\vec{\nabla}\cdot\vec{J}_p = -\frac{\partial\rho_b}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{P} = \nabla\cdot\left(\frac{\partial\vec{P}}{\partial t}\right)$$

$$\Rightarrow \vec{J}_p = \frac{\partial\vec{P}}{\partial t}$$

$$\Rightarrow \vec{J}_b = \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right) \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial\vec{P}}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

define $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

$\Rightarrow \quad \vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$ inhomogeneous eqs.

$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ homogeneous eqs.

for linear materials, $\left.\begin{array}{l}\vec{D} = \varepsilon\vec{E} \\ \vec{H} = \frac{1}{\mu}\vec{B}\end{array}\right\}$ closes above equations.

If we had $\vec{D}(r,t) = \varepsilon E(\vec{r},t)$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu} B(\vec{r},t)$$

then Maxwell's eq'ns, in absense of free charge + free current would be

$$\varepsilon \vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

everything would be the same except $\varepsilon_0\mu_0 \to \varepsilon\mu > \varepsilon_0\mu_0$

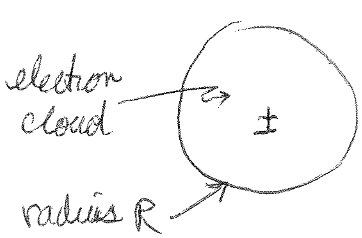
The speed of EM waves in the material would be

$$v = \frac{1}{\sqrt{\varepsilon\mu}} < c \qquad c/v \equiv n \quad \text{index of refraction}$$

would have $|\vec{B}| = \frac{1}{v}|\vec{E}|$

In general however, things are <u>much more complicated</u> for time varying response

Consider model for polarization of a neutral atom, that we saw last semester



If displace center of electron cloud from ion by distance $\vec{r}$, then there is a restoring force

$$\vec{F}_{rest} = \frac{-e^2\vec{r}}{4\pi\varepsilon_0 R^3} \equiv -m\omega_0^2\vec{r}$$

electron mass

ω_0 has units of frequency.