If we had $\vec{D}(\vec{r},t) = \varepsilon E(\vec{r},t)$

$\vec{H}(\vec{r},t) = \frac{1}{\mu} B(\vec{r},t)$

then Maxwell's eq'ns, in absense of free charge + free current would be

$$\varepsilon \vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

everything would be the same except $\varepsilon_0\mu_0 \rightarrow \varepsilon\mu > \varepsilon_0\mu_0$

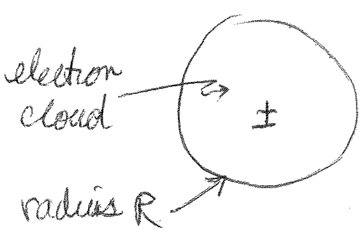
the speed of EM waves in the material would be

$$v = \frac{1}{\sqrt{\varepsilon\mu}} < c \qquad c/v \equiv n \text{ index of refraction}$$

would have $|\vec{B}| = \frac{1}{v}|\vec{E}|$

In general however, things are <u>much more complicated</u> for time varying response

Consider model for polarization of a neutral atom, that we saw last semester



If displace center of electron cloud from ion by distance $\vec{r}$, then there is a restoring force

$$\vec{F}_{rest} = \frac{-e^2\vec{r}}{4\pi\varepsilon_0 R^3} \equiv -m\omega_0^2\vec{r}$$

electron mass

ω_0 has units of frequency.

(electric field from electron cloud increases linearly with distance from origin)

Also in general will be a damping, or friction force, due to energy transfer from atom to other degrees of freedom

$$\vec{F}_{damp} = -m\gamma \frac{d\vec{r}}{dt} \qquad \text{friction} \sim \text{velocity}$$

If electron is in external electric field, the eqn of motion is then

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{tot} = -e\vec{E}(t) - m\omega_0^2\vec{r} - m\gamma\frac{d\vec{r}}{dt}$$

or $$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + \omega_0^2\vec{r} = -\frac{e\vec{E}(t)}{m}$$

(assuming that $\vec{E}$ is constant over spatial distances that electron moves)

Consider sinusoidal $\vec{E}$ field, in complex form.

$$\vec{E}(t) = \vec{E}_\omega e^{-i\omega t}$$

Assume solution $\vec{r}(t) = \vec{r}_\omega e^{-i\omega t}$

$$-\omega^2\vec{r}_\omega - i\omega\gamma\vec{r}_\omega + \omega_0^2\vec{r}_\omega = -\frac{e\vec{E}_\omega}{m}$$

$$\vec{r}_\omega = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_\omega$$

dipole moment $\vec{p}(t) = -e\vec{r}(t) = \vec{p}_\omega e^{-i\omega t}$

$$\vec{p}_\omega = \frac{e^2}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)}\vec{E}_\omega$$

↑ resonance at $\omega \approx \omega_0$
width of resonance is γ

frequency dependent polarizability

$\vec{p}_\omega = \alpha(\omega)\vec{E}_\omega \qquad \alpha(\omega) = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Since $\alpha(\omega)$ is <u>complex</u> the polarization does not in general oscillate in phase with the electric field.

For a pure harmonic, $\vec{p}(t) = \alpha(\omega)\vec{E}_\omega e^{-i\omega t}$

$= |\alpha(\omega)|\vec{E}_\omega e^{-i(\omega t - \delta)}$ phase shift from $\vec{E}$

where δ is the phase of complex α

ie $\alpha = |\alpha| e^{i\delta}$

For a general electric field $\vec{E}(t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega e^{-i\omega t}$

the response is $\vec{p}(t) = \int_{-\infty}^{\infty} d\omega\, \vec{p}_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega\, \alpha(\omega)\vec{E}_\omega e^{-i\omega t}$

substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) e^{-i\omega(t-t')}$$

define Fourier transform $\tilde{\alpha}(t) \equiv \int_{-\infty}^{\infty} d\omega\, \alpha(\omega) e^{-i\omega t}$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t')\tilde{\alpha}(t-t')$$

$\vec{p}$ at t is due to $\vec{E}$ at all other times t', not only at time t

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<u>True in general</u>: if $\tilde{A}(\omega) = \tilde{\alpha}(\omega)\tilde{B}(\omega)$ is relation between Fourier transforms, then in time,

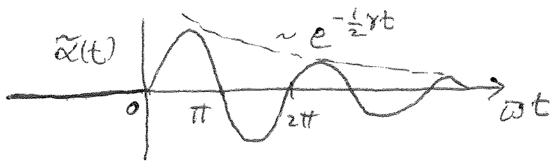
$$A(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} B(t')\, a(t-t')$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi}\, \vec{E}(t')\, \tilde{\alpha}(t-t')$$ response is <u>non-local</u> in time

i.e. $\vec{p}(t)$ is determined not just by the instaneous $\vec{E}(t)$, but by $\vec{E}(t')$ at other times $t' \neq t$.

For our simple model, $\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

$$\tilde{\alpha}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \alpha(\omega) = \begin{cases} \frac{e^2}{m} \frac{1}{\bar{\omega}} e^{-\frac{1}{2}\gamma t} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



where $\bar{\omega} = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$

$\tilde{\alpha}(t) = 0$ for $t < 0 \Rightarrow$ causal response, i.e. $\vec{p}(t)$ depends on $\vec{E}(t')$ only for <u>earlier</u> times $t' < t$

$\tilde{\alpha}(t)$ gives the polarization that results from a δ-function pulse in $\vec{E}$ at time $t' = 0$, i.e. $\vec{E}(t') = \vec{E}_0\, \delta(t')$

$\tilde{\alpha}(t)$ has the familiar form of the displacement of a damped harmonic oscillator that is given an impulse kick at $t' = 0$

For a dielectric, we now expect <u>polarization density</u> from a pure sinusoidal electric field, will be

$\vec{\mathbb{P}}(t) = \vec{\mathbb{P}}_\omega e^{-i\omega t}$ with $\vec{\mathbb{P}}_\omega = \varepsilon_0 \chi_e(\omega) \vec{E}_\omega$

$\chi_e(\omega)$ freq dependent electric susceptibility

where $\chi_e(\omega) \approx \frac{N \alpha(\omega)}{\varepsilon_0}$ for a dilute density of atoms

N = atomic density
i.e. atoms per volume

$\Rightarrow$ Displacement $\vec{D}(t) = \vec{D}_\omega e^{-i\omega t}$ with $\vec{D}_\omega = \varepsilon_0 \vec{E}_\omega + \vec{P}_\omega$

$$= \varepsilon_0 (1 + \chi_e(\omega)) \vec{E}_\omega$$

$$= \varepsilon(\omega) \vec{E}_\omega$$

$$\varepsilon(\omega) = \varepsilon_0 (1 + \chi_e(\omega))$$

$$= \varepsilon_0 K(\omega)$$

complex, freq dependent permeativity

freq dependent electric susceptibility

$\chi_e(\omega) \approx \frac{N\alpha(\omega)}{\varepsilon_0}$ ← freq dependent atomic polarizability

freq dependent dielectric function

$\vec{D}_\omega = \varepsilon(\omega) \vec{E}_\omega \Rightarrow \vec{D}(t) \neq \varepsilon \vec{E}(t)$, but rather

ε complex $\Rightarrow \vec{D}_\omega$ and $\vec{E}_\omega$ are not in general in phase with each other

$$\vec{D}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \tilde{\varepsilon}(t - t')$$

↖ F.T. of $\varepsilon(\omega)$

$\vec{D}(t)$ and $\vec{E}(t)$ are <u>non-locally</u> related in time

$\Rightarrow$ Maxwell's equations look very complicated when expressed in terms of time. For example:

assume $\mu = \mu_0$, $\vec{J}_{free} = 0$, then Ampere's Law is

$$\cancel{\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \frac{\partial \vec{D}}{\partial t} = \int \frac{dt'}{2\pi} \vec{E}(t')}$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B}(\vec{r}, t) = \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(\vec{r}, t') \frac{d}{dt} \tilde{\varepsilon}(t - t')$$

becomes an integro-differential equation when expressed in terms of $\vec{B}$ and $\vec{E}$.

(Alternatively, Faraday's Law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, would become an integro-differential equation if expressed in terms of $\vec{H}$ and $\vec{D}$.)

Does $\vec{B}$ in material solve wave equation?

For $\mu = \mu_0$, $H = \frac{B}{\mu_0}$ $\quad \rho_f = \vec{J}_f = 0$

Amperes $\quad \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} = \mu_0 \vec{\nabla} \times \frac{\partial \vec{D}}{\partial t} = \int \frac{dt'}{2\pi} (\vec{\nabla} \times \vec{E}(r,t')) \frac{d}{dt} \tilde{\varepsilon}(t-t')$$

$$= -\mu_0 \int \frac{dt'}{2\pi} \frac{\partial \vec{B}(t')}{\partial t'} \frac{d}{dt} \tilde{\varepsilon}(t-t')$$

use $\frac{d\tilde{\varepsilon}(t-t')}{dt} = -\frac{d\tilde{\varepsilon}(t-t')}{dt'}$

$$= +\mu_0 \int \frac{dt'}{2\pi} \frac{\partial \vec{B}}{\partial t'} \frac{d\tilde{\varepsilon}}{dt'}(t-t')$$

integrate by parts

$$= -\mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 \vec{B}}{\partial t'^2} \tilde{\varepsilon}(t-t')$$

$$\nabla^2 \vec{B}(\vec{r},t) - \mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 \vec{B}(\vec{r},t')}{\partial t'^2} \tilde{\varepsilon}(t-t') = 0$$

This is not in general the wave equation!

However If $\varepsilon(\omega) = \frac{\varepsilon}{2\pi}$ const, then ie $\varepsilon(\omega)$ indep of freq

Thus $\tilde{\varepsilon}(t) = \int d\omega \, e^{-i\omega t} \varepsilon(\omega) = \varepsilon \, \delta(t)$ local in time

$\tilde{\varepsilon}(t-t') = \varepsilon \, \delta(t-t')$ $\quad$ recover wave equation

$$\nabla^2 \vec{B} - \mu_0 \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \qquad v = \frac{1}{\sqrt{\mu_0 \varepsilon}}$$