Maxwell's Equs only look simple when expressed in terms of Fourier Transforms.

For pure sinusoidal solutions:

\[
\begin{align*}
\vec{E}(\vec{r}, t) &= \vec{E}_0 \, e^{i(k \cdot \vec{r} - wt)} \\
\vec{B}(\vec{r}, t) &= \vec{B}_0 \, e^{i(k \cdot \vec{r} - wt)} \\
\vec{H}(\vec{r}, t) &= \vec{H}_0 \, e^{i(k \cdot \vec{r} - wt)} \\
\vec{D}(\vec{r}, t) &= \vec{D}_0 \, e^{i(k \cdot \vec{r} - wt)}
\end{align*}
\]

for EM waves in dielectric, assume \( \vec{J}_f = \vec{E}_f = 0 \)

Maxwell's Equs:

\[ \nabla \cdot \vec{D} = 0, \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \]

Assume \( \mu = \mu_0 \Rightarrow \vec{H}_0 = \frac{1}{\mu_0} \vec{B}_0 \)

dielectric response given by \( \varepsilon(\omega) \Rightarrow \vec{D}_0 = \varepsilon(\omega) \vec{E}_0 \)

For \( \vec{J}_f = \vec{E}_f = 0 \), Maxwell's Equs in terms of the Fourier amplitudes are then:

1. \[ i \vec{k} \cdot \vec{D}_0 = i \varepsilon(\omega) \vec{k} \cdot \vec{E}_0 = 0 \Rightarrow \vec{k} \cdot \vec{E}_0 = 0 \]
2. \[ i \vec{k} \cdot \vec{B}_0 = 0 \]
3. Faraday \[ i \vec{k} \times \vec{E}_0 = i \omega \vec{B}_0 \]
4. Ampere \[ i \vec{k} \times \vec{H}_0 = -i \omega \vec{D}_0 \Rightarrow i \vec{k} \times \vec{B}_0 = -i \omega \varepsilon(\omega) \vec{E}_0 \]

\[ \vec{k} \times (\text{Faraday}) = i \vec{k} \times (\vec{k} \times \vec{E}_0) = i \omega (\vec{k} \times \vec{B}_0) \quad \text{substitute in from Ampere} \]

\[ i \vec{k} \times (\vec{k} \times \vec{E}_0) = \vec{k} (\vec{k} \cdot \vec{E}_0) - \vec{E}_0 (\vec{k} \cdot \vec{k}) = -i \omega^2 \varepsilon(\omega) \mu_0 \vec{E}_0 \]

\[ = 0 \text{ by (1))} \]

\[ \Rightarrow k^2 \vec{E}_0 = \omega^2 \varepsilon(\omega) \mu_0 \vec{E}_0 \]
\[ k^2 = \frac{\mu_0}{\varepsilon_0} \varepsilon(\omega) \]

Use \( \frac{1}{c^2} = \mu_0 \varepsilon_0 \)

"Dispersion" relation for waves in dielectric.

Dispersion relation determines wave vector \( k \), for a given frequency \( \omega \).

Note \( \frac{\omega^2}{k^2} \neq \text{constant} \Rightarrow \vec{E} \) is not solution of a wave equation \( \nabla^2 \vec{E} = 0 \) for different frequencies travel with different speeds.

Since \( \varepsilon(\omega) \) is complex, \( \varepsilon(\omega) = \varepsilon_1(\omega) + i \varepsilon_2(\omega) \)

\[ \begin{bmatrix} \text{Re}[\varepsilon] \\ \text{Im}[\varepsilon] \end{bmatrix} \]

then in general the wavevector is also complex

\[ k = k_1 + i k_2 = \pm \omega \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} + i \frac{\varepsilon_2}{\varepsilon_0} \]

For a wave traveling in \( \hat{z} \) direction, \( k = k \hat{z} \), we have

\[ \vec{E}(\vec{r},t) = E_0 e^{i(k \cdot \vec{r} - \omega t)} = E_0 e^{i(k_1 + i k_2) \cdot \vec{r} - \omega t} = E_0 e^{-k_2 y} e^{i(k_1 x - \omega t)} \]

If choose \( + \sqrt{\cdot} \) solution for \( k_1 \), so that wave propagates in \( + \hat{z} \) direction, then should take \( + \sqrt{\cdot} \) solution for \( k_2 \), so that wave decays as it propagates into material.
Decay length = \( \frac{1}{k_2} \)  
\( k_2 \) is called the attenuation.

Since intensity is \( \propto E^2 \) decays as \( e^{-2k_2z} \), \( 2k_2 \) is called the absorption coefficient.

Physical origin of decay: EM wave excites atom to oscillate. Oscillations pump energy into other degrees of freedom, due to changing \( \gamma \). \( \Rightarrow \) EM wave is pumping energy into material \( \Rightarrow \) Energy contained in EM wave should decrease as it propagates into material \( \Rightarrow \) Ablative decay.

Phase velocity of wave \( v_p = \frac{\omega}{k_1} \) depends on frequency.

Index of refraction \( n = \frac{c}{v_p} = \frac{c}{\omega/k_1} \) depends on freq.

Let's look now at magnetic field. From Faraday:

\[
\mathbf{B}_\omega = \frac{k}{\omega} \times \mathbf{E}_\omega = \left( \frac{k_1 + ik_2}{\omega} \right) \hat{z} \times \mathbf{E}_\omega
\]

Write \( k_1 + ik_2 = \sqrt{k_1^2 + k_2^2} e^{i\delta} \) where \( \delta = \arctan \left( \frac{k_2}{k_1} \right) \) is phase of \( k \).

\[
\mathbf{B}_\omega = \frac{|k|}{\omega} \hat{z} \times \mathbf{E}_\omega e^{i\delta}
\]
\[ \vec{B}(\vec{r}, t) = \frac{|k|}{\omega} \left( \hat{z} \times \vec{E}_0 \right) e^{i(k \cdot \vec{r} - \omega t + \delta)} \]

\[ = \frac{|k|}{\omega} \left( \hat{z} \times \vec{E}_0 \right) e^{-k_2 \gamma} e^{i(k_2 \gamma - \omega t + \delta)} \]

**Physical fields:**

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 e^{-k_2 \gamma} \cos(k_2 \gamma - \omega t) \]

\[ \vec{B}(\vec{r}, t) = (\hat{z} \times \vec{E}_0) \frac{|k|}{\omega} e^{-k_2 \gamma} \cos(k_2 \gamma - \omega t + \delta) \]

1. \( \vec{E} \) and \( \vec{B} \) are transverse to \( \hat{k} \), and \( \vec{E} \perp \vec{B} \)

2. Ratio of amplitudes

\[ \frac{|\vec{B}|}{|\vec{E}|} = \frac{|k_2|}{\omega} = \frac{\sqrt{k_1^2 + k_2^2}}{\omega} = \frac{\sqrt{\varepsilon(\omega)}}{\varepsilon_0} \]

3. \( \vec{B} \) wave is shifted with respect to \( \vec{E} \) wave by

   phase shift \( \delta = \arctan(\frac{k_2}{k_1}) \) (see Fig. 8.21 in text)

**Summary**

**Main consequences of complex \( \varepsilon(\omega) \):**

1. Waves decay as they propagate \( \sim e^{-k_2 \gamma} \)

2. \( \vec{E} \) and \( \vec{B} \) waves shifted in phase by \( \delta = \arctan(\frac{k_2}{k_1}) \)

   (And if \( \varepsilon > 0 \))

   If \( \varepsilon_2 = \text{Im}[\varepsilon(\omega)] = 0 \), then \( \varepsilon \) real, \( k \) real, \( k_2 = 0 \)

   \( \Rightarrow \) no decay and no phase shift.

**Main consequences of frequency dependent \( \varepsilon(\omega) \):**

1. \( \vec{E}(t) \) and \( \vec{B}(t) \) non-locally related in time

2. Waves of different \( \omega \) travel with different velocities

   \( v_p = \frac{\omega}{k_1} \)

3. **Dispersion** - wave pulses do not travel with \( v_p \),
   and do not keep their shape as they propagate.
Phase velocity and group velocity and dispersion

$$k^2 = \frac{\omega^2}{c^2} \frac{\varepsilon(\omega)}{\varepsilon_0}$$

For simplicity, assume $\varepsilon(\omega)$ is real and positive.

$$k = \frac{\omega}{c} \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon(\omega)/\varepsilon_0}} = \frac{c}{\sqrt{k(\omega)}}$$

Index of refraction

$$n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{k(\omega)}$$

Sinusoidal waves $e^{i(k\cdot -\omega t)}$ propagate with different phase speeds $v_p(\omega)$ for different $\omega$.

$v_p$ is speed with which peaks in oscillation move to right.

If take linear superposition of many sinusoidal waves, then each different freq $\omega$, moves with different speed $v_p(\omega)$. So the shape of the wave is not preserved in time.

This is another way to see that waves in a dielectric do not solve the wave equation - for the wave equation, all freq move with same speed $c$ indep of $\omega$, and the shape of the wave is always preserved in time, as solutions are always of form $f(k\cdot -\omega t)$.