$\Longrightarrow$ Maxwell's Equ only look simple when expressed in terms of Fourier Transforms.

For pure sinusoidal solutions:
$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{B}(\vec{r},t) = \vec{B}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$
$$\vec{D}(\vec{r},t) = \vec{D}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

for EM waves in dielectric, assume $\rho_f = \vec{J}_f = 0$

Maxwell's Equs: $\vec{\nabla}\cdot\vec{D} = 0$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$

assume $\mu = \mu_0 \Rightarrow \vec{H}_\omega = \frac{1}{\mu_0}\vec{B}_\omega$

dielectric response given by $\varepsilon(\omega) \Rightarrow \vec{D}_\omega = \varepsilon(\omega)\vec{E}_\omega$

For $\rho_f = \vec{J}_f = 0$, Maxwell's Equs in terms of the Fourier amplitudes are then

1) $i\vec{k}\cdot\vec{D}_\omega = i\varepsilon(\omega)\vec{k}\cdot\vec{E}_\omega = 0 \quad \rightarrow \vec{k}\cdot\vec{E}_\omega = 0 \quad \vec{k}\perp\vec{E}_\omega$

2) $i\vec{k}\cdot\vec{B}_\omega = 0 \quad \rightarrow \vec{k}\cdot\vec{B}_\omega = 0 \quad \vec{k}\perp\vec{B}_\omega$

transverse

3) Faraday $\quad i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega$

4) Ampere $\quad i\vec{k}\times\vec{H}_\omega = -i\omega\vec{D}_\omega \Rightarrow \frac{i\vec{k}}{\mu_0}\times\vec{B}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$

$$\vec{k}\times(\text{Faraday}) = i\vec{k}\times(\vec{k}\times\vec{E}_\omega) = i\omega(\vec{k}\times\vec{B}_\omega) = -i\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

substitute in from Ampere

$$\vec{k}\times(\vec{k}\times\vec{E}_\omega) = \vec{k}\underbrace{(\vec{k}\cdot\vec{E}_\omega)}_{=0 \text{ by (1)}} - \vec{E}_\omega(\vec{k}\cdot\vec{k}) = -\omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Longrightarrow k^2\vec{E}_\omega = \omega^2\varepsilon(\omega)\mu_0\vec{E}_\omega$$

$$\Rightarrow \boxed{\begin{aligned} k^2 &= \omega^2 \varepsilon(\omega)\mu_0 \\ k^2 &= \frac{\omega^2}{c^2}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) \end{aligned}}$$

use $\frac{1}{c^2} = \mu_0 \varepsilon_0$

"dispersion" relation for waves in dielectric

dispersion relation determines wave vector k, for a given frequency ω.

Note $\frac{\omega^2}{k^2} \neq$ constant $\Rightarrow \vec{E}$ is not solution of a wave equation $\Box^2 \vec{E} = 0$. different frequencies travel with different speeds.

Since $\varepsilon(\omega)$ is complex $\quad \varepsilon(\omega) = \underbrace{\varepsilon_1(\omega)}_{Re[\varepsilon]} + i \underbrace{\varepsilon_2(\omega)}_{Im[\varepsilon]}$

then in general the wavevector is also complex

$$k = k_1 + i k_2 = \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i\frac{\varepsilon_2}{\varepsilon_0}}$$

For a wave traveling in $\hat{z}$ direction, $\vec{k} = k\hat{z}$, we have

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{i((k_1 + ik_2)z - \omega t)}$$

$$= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

If choose $+\sqrt{\ }$ solution for k_1, so that wave propagates in $+\hat{z}$ direction, then should take $+\sqrt{\ }$ solution for k_2, so that wave decays as it propagates into material

$\underline{\text{decay length}} = 1/k_2$ $\quad$ k_2 is called the $\underline{\text{attenuation}}$

Since intensity is $\sim E^2$ decays as $e^{-2k_2 z}$, $2k_2$ is called the $\underline{\text{absorption coefficient}}$

$\underline{\text{physical origin of decay}}$: EM wave excits atom to oscillate. Oscillations pump energy into other degrees of freedom, due to damping γ. $\Rightarrow$ EM wave is pumping energy into material $\Rightarrow$ Energy contained in EM wave should decrease as it propagates into material $\Rightarrow$ amplitude decays.

$\underline{\text{phase velocity of wave}}$ $\quad v_p \equiv \dfrac{\omega}{k_1}$ $\quad$ depends on frequency

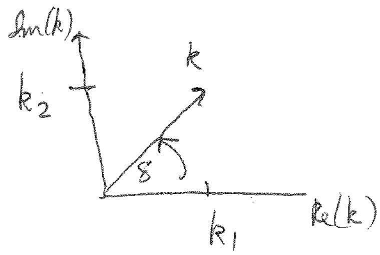
$\underline{\text{index of refraction}}$ $\quad n \equiv \dfrac{c}{v_p} = \dfrac{c k_1}{\omega}$ $\quad$ depends on freq

Lets look now at magnetic field. From Faraday

$$\vec{B}_\omega = \frac{\vec{k}}{\omega} \times \vec{E}_\omega = \frac{(k_1 + i k_2)}{\omega} \hat{z} \times \vec{E}_\omega$$

write $k_1 + i k_2 = \sqrt{k_1^2 + k_2^2}\, e^{i\delta}$ $\quad$ where $\delta = \arctan\left(\dfrac{k_2}{k_1}\right)$

$= |k| e^{i\delta}$ $\quad$ is phase of k



$$\vec{B}_\omega = \frac{|k|}{\omega} \hat{z} \times \vec{E}_\omega e^{i\delta}$$

$$\vec{B}(\vec{r},t) = \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)}$$

$$= \frac{|k|}{\omega}(\hat{z}\times\vec{E}_\omega)\, e^{-k_2 z}\, e^{i(k_1 z-\omega t+\delta)}$$

<u>Physical fields</u> :

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2 z}\cos(k_1 z-\omega t)$$

$$\vec{B}(\vec{r},t) = (\hat{z}\times\vec{E}_\omega)\,\frac{|k|}{\omega}\, e^{-k_2 z}\cos(k_1 z-\omega t+\delta)$$

⇒ (1) $\vec{E}$ and $\vec{B}$ are transverse to $\vec{k}$, and $\vec{E}\perp\vec{B}$

(2) ratio of amplitudes $\frac{|\vec{B}|}{|\vec{E}|} = \frac{|k|}{\omega} = \frac{\sqrt{k_1^2+k_2^2}}{\omega} = \sqrt{\left|\frac{\varepsilon(\omega)}{\varepsilon_0}\right|}\,\frac{1}{c}$

(3) $\vec{B}$ wave is shifted with respect to $\vec{E}$ wave by phase shift $\delta = \arctan(k_2/k_1)$ (see Fig 8.21 in text)

<u>Summary</u>

Main consequenses of <u>complex</u> $\varepsilon(\omega)$

1) Waves decay as they propagate $\sim e^{-k_2 z}$

2) $\vec{E}$ and $\vec{B}$ waves shifted in phase by $\delta = \arctan(k_2/k_1)$

If $\varepsilon_2 = \mathrm{Im}[\varepsilon(\omega)] = 0$, then $\underbrace{\varepsilon \text{ real},}_{\text{and if } \varepsilon_1>0}$ ⇒ k real, $k_2 = 0$

⇒ no decay and no phase shift.

Main consequenses of <u>freq dependent</u> $\varepsilon(\omega)$

(1) $\vec{E}(t)$ and $\vec{D}(t)$ non-locally related in time

(2) waves of different ω travel with different velocities $v_p = \frac{\omega}{k_1}$

(3) <u>dispersion</u> – wave pulses do not travel with v_p, and do not keep their shape as they propagate

Phase velocity and group velocity and dispersion

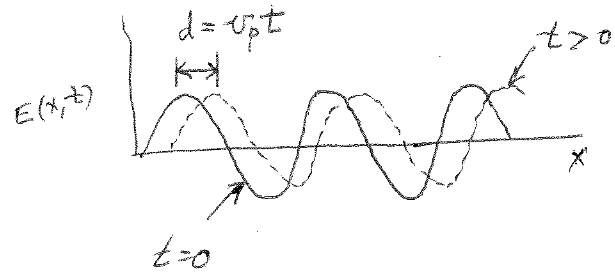
$k^2 = \frac{\omega^2}{c^2} \frac{\varepsilon(\omega)}{\varepsilon_0}$ For simplicity, assume $\varepsilon(\omega)$ is real and positive

$k = \frac{\omega}{c} \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$ $v_p = \frac{\omega}{k} = c\sqrt{\frac{\varepsilon_0}{\varepsilon(\omega)}} = \frac{c}{n}$

index of refraction $n(\omega) = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} = \sqrt{K(\omega)}$

sinusoidal waves $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ propagate with different phase speeds $v_p(\omega)$ for different ω.

v_p is speed with which peaks in oscillation move to right



for $\vec{E} = E e^{i(kx - \omega t)}$
with $\omega = v_p(\omega) k$

If take linear superposition of many sinusoidal waves, then each different freq ω, moves with different speed $v_p(\omega)$. So the shape of the wave is <u>not</u> preserved in time.
[This is another way to see that waves in a dielectric do not solve the wave equation - for the wave equation, all freq move with same speed v indep of ω, and the shape of the wave is always preserved in time, i.e. solutions are always of form $g(\vec{k}\cdot\vec{r} - \omega t)$]