Consider a superposition

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)} \qquad k(\omega) = \frac{\omega}{c}\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

At $\vec{r}=0$, $\vec{E}(0,t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega t}$ so $\vec{E}_\omega$ is F.T. of $\vec{E}(0,t)$

At some position $\vec{r} \neq 0$

$$\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{i(k(\omega)z - \omega t)}$$

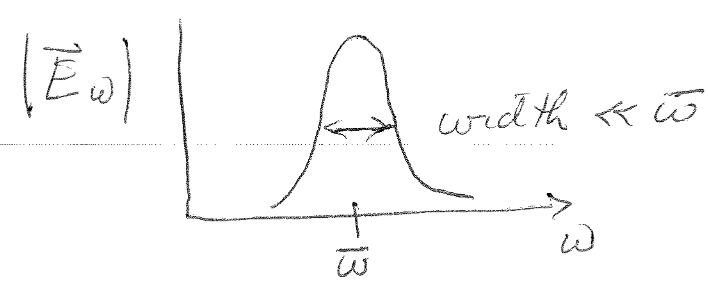
if $\underline{\underline{\text{no}}}$ dispersion, ie $k = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \frac{\omega}{v_p}$ with v_p indep of ω

then $\vec{E}(\vec{r},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - z/v_p)}$

$$= \vec{E}(0, t - z/v_p)$$ ← form of solution to wave equation

field at z at time t, is same field as was at $z=0$ at the earlier time $t - z/v_p$ ⇒ wave moved distance z in time z/v_p ⇒ speed of wave is v_p

Suppose now that $\varepsilon(\omega)$ <u>does</u> depend on ω, so there is dispersion. Suppose $\vec{E}_\omega$ is strongly peaked about some average $\bar{\omega}$



then $k(\omega) \cong k(\bar{\omega}) + \left.\frac{dk}{d\omega}\right|_{\bar{\omega}} (\omega - \bar{\omega}) + \ldots$

$$\vec{E}(\vec{r},t) = \int d\omega\, \vec{E}_\omega\, e^{i(k(\bar{\omega})z + \frac{dk}{d\omega}\omega z - \frac{dk}{d\omega}\bar{\omega} z - \omega t)}$$

$$= e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z} \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega\, e^{-i\omega(t - \frac{dk}{d\omega} z)}$$

$$= \underbrace{e^{i(k(\bar{\omega}) - \frac{dk}{d\omega}\bar{\omega})z}}_{\text{phase factor}} \; \underbrace{\vec{E}(0, t - \tfrac{dk}{d\omega} z)}_{\text{envelope - determines shape of pulse}}$$

intensity of wave $\sim |\vec{E}|^2$

$$|\vec{E}|^2(\vec{r},t) = |\vec{E}|^2(0, t - \tfrac{dk}{d\omega} z)$$

intensity travels with velocity $v_g = 1/\left(\frac{dk}{d\omega}\right)_{\bar{\omega}} = \frac{d\omega}{dk}$ = <u>group velocity</u>

<u>not</u> with average phase velocity $\bar{v}_p = \frac{\bar{\omega}}{k(\bar{\omega})}$

only when $\varepsilon(\omega)$ is indep of ω will $v_p = v_g$

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega}\left[\frac{\omega}{c} n(\omega)\right] = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p}{c}\omega\frac{dn}{d\omega}} \Rightarrow \text{when } \frac{dn}{d\omega} > 0, \; v_g < v_p \quad (1)$$

$$\text{when } \frac{dn}{d\omega} < 0, \; v_g > v_p \quad (2)$$

case (1) is called "normal" dispersion

case (2) is called "anomalous" dispersion

Our result $\vec{E}^2(r,t) = \vec{E}^2(0, t - \frac{dk}{d\omega} z)$ looks like we still preserve shape of wave – but this is due to the simplicity of our approximation. If we kept to next order, i.e. used $k(\omega) = k(\bar{\omega}) + \frac{dk}{d\omega}(\omega - \bar{\omega})$

$$+ \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\bar{\omega})^2$$

one would find that the wave pulse changes shape as it propagates – in particular, it spreads.

A simple way to estimate this effect:

If pulse initially has width $\Delta\omega$ about $\bar{\omega}$, i.e. $\vec{E}_\omega$ looks like

there is a spread in group velocities

$$\Delta v_g \simeq \left|\frac{dv_g}{d\omega}\right| \Delta\omega = \left|\frac{d}{d\omega}\left(\frac{1}{dk/d\omega}\right)\right| \Delta\omega$$

$$= \frac{1}{(dk/d\omega)^2}\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega = v_g^2 \left|\frac{d^2k}{d\omega^2}\right|\Delta\omega$$

So if pulse take a time $T = z/v_g$ to reach point z from the origin, there is also a spread in arrival times

$$\Delta T = \Delta(z/v_g) = \frac{z}{v_g^2}\Delta v_g = z\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega$$

ΔT gives a spreading of width of the wave pulse, that grows linearly with the distance z traveled.

For a pulse of width $\Delta\omega$, the width in time is

$$\Delta t \sim \frac{1}{\Delta\omega} \quad \text{(like uncertainty principle in QM)}$$

$$\Rightarrow \Delta T \simeq z \left| \frac{d^2k}{d\omega^2} \right| \frac{1}{\Delta t}$$

$\Rightarrow$ the sharper the pulse is initially, (ie the smaller Δt) the faster it spreads as it travels (ie the larger ΔT is).

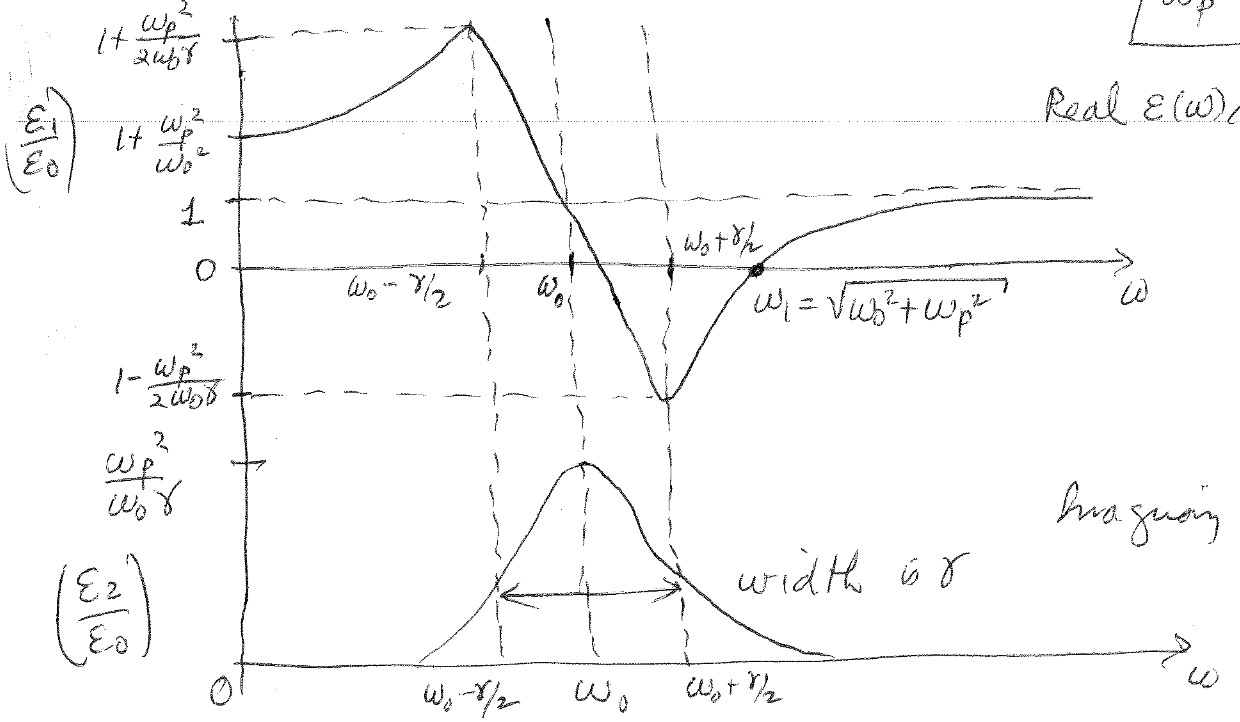
For our simple model

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$$

$$\Rightarrow \quad \frac{\varepsilon_1}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$ real part ε

$$\frac{\varepsilon_2}{\varepsilon_0} = \frac{Ne^2}{m\varepsilon_0}\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$ Imaginary part ε

$$\boxed{\omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0}}$$ plasma freq



as $\left(\frac{\gamma}{\omega_0}\right)\to 0$, width of resonance decreases
height of peaks diverges

Notes for sketch $\varepsilon_1/\varepsilon_0$

max and min of $\varepsilon_1/\varepsilon_0$ occur when $\frac{\partial(\varepsilon_1/\varepsilon_0)}{\partial\omega} = 0$

$$\Rightarrow \left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right](-2\omega) - (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)(-2\omega)+2\omega\gamma^2\right] = 0$$

$$(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2 - 2(\omega_0^2-\omega^2)^2 + (\omega_0^2-\omega^2)\gamma^2 = 0$$

$$(\omega_0^2-\omega^2)^2 = \omega_0^2\gamma^2$$
$$|\omega_0^2-\omega^2| = \omega_0\gamma$$
$$|\omega_0-\omega|(\omega_0+\omega) = \omega_0\gamma$$

for sharp resonance, peaks are when $\frac{\omega-\omega_0}{\omega_0} \ll 1 \rightarrow \omega_0+\omega \approx 2\omega_0$

$$\Rightarrow |\omega_0-\omega|\, 2\omega_0 = \omega_0\gamma$$
$$|\omega_0-\omega| = \frac{\gamma}{2} \Rightarrow \boxed{\omega-\omega_0 = \pm\frac{\gamma}{2}}$$ location of max and min

$\Rightarrow$ width of resonance $= \gamma$

zero's of ε_1 define $\omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0}$

$$0 = 1 + \omega_p^2 \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$\Rightarrow (\omega^2-\omega_0^2)^2 - \omega_p^2(\omega^2-\omega_0^2) + \omega^2\gamma^2 = 0$$

For the zero near the resonance, $\omega^2\gamma^2 \rightarrow \omega_0^2\gamma^2$ a good approx

$\omega^2-\omega_0^2 \rightarrow (\Delta\omega)2\omega_0$, $\Delta\omega \equiv \omega-\omega_0$

$$(\Delta\omega)^2 4\omega_0^2 - \Delta\omega\, 2\omega_0\omega_p^2 + \omega_0^2\gamma^2 = 0$$
$$(\Delta\omega)^2 - \frac{\omega_p^2}{2\omega_0}\Delta\omega + \frac{\gamma^2}{4} = 0$$

for $\omega_p \gg \omega_0$, $\Delta\omega \simeq \frac{\gamma^2\omega_0}{2\omega_p^2} = \frac{\gamma}{2}\underbrace{\left(\frac{\gamma}{\omega_0}\right)\left(\frac{\omega_0}{\omega_p}\right)^2}_{\text{both small}}$

generally true

shift of resonance small compared to width of resonance

For the zero above the resonance at ω_1

$$(\omega_1^2-\omega_0^2)^2 - \omega_p^2(\omega_1^2-\omega_0^2) + \omega_1^2\gamma^2 = 0$$

small so ignore

$$\Rightarrow \quad \omega_1^2-\omega_0^2 = \omega_p^2$$

$$\omega_1^2 = \omega_0^2+\omega_p^2 \simeq \omega_p^2 \quad \text{when } \omega_p \gg \omega_0$$

<u>max of ~~___~~</u> ε_2

$$\varepsilon_2 = \omega_p^2 \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

peak when $\frac{\partial \varepsilon_2}{\partial \omega} = 0 \Rightarrow ((\omega_0^2-\omega^2)^2+\omega^2\gamma^2)\gamma - \omega\gamma[2(\omega_0^2-\omega^2)(-2\omega) + 2\omega\gamma^2] = 0$

$$\Rightarrow (\omega_0^2-\omega^2)^2\gamma + 4\omega^2\gamma(\omega_0^2-\omega^2) - \omega^2\gamma^3 = 0$$

near resonance,

$$(\omega_0^2-\omega^2) = \Delta\omega(2\omega_0) = \frac{\omega^2\gamma^3}{4\omega^2\gamma} = \frac{\gamma^2}{4}$$

$$\Delta\omega = \frac{\gamma^2}{8\omega_0} \quad \text{small} \Rightarrow \text{peak at} \approx \omega_0$$

$$\frac{\varepsilon_2}{\varepsilon_0}(\omega_0) = \frac{\omega_p^2}{\omega\gamma}$$

half height at ω such that $\frac{\varepsilon_2}{\varepsilon_0}(\omega) = \frac{\omega_p^2}{2\omega\gamma}$

$$\Rightarrow \quad \frac{1}{2\omega\gamma} = \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \Rightarrow (\omega_0^2-\omega^2)^2+\omega^2\gamma^2 = 2\omega^2\gamma^2$$

$$\omega_0^2-\omega^2 = \pm\omega\gamma$$

for <u>sharp</u> resonance $\Delta\omega(2\omega_0) = \pm\omega_0\gamma$

$$\Delta\omega \simeq \pm\frac{\gamma}{2}$$

width of resonance peak in $\frac{\varepsilon_2}{\varepsilon_0}$ is γ.

$$k = k_1 + i k_2 = \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i \frac{\varepsilon_2}{\varepsilon_0}}$$

want to express k_1 and k_2 in terms of ε_1 and ε_2

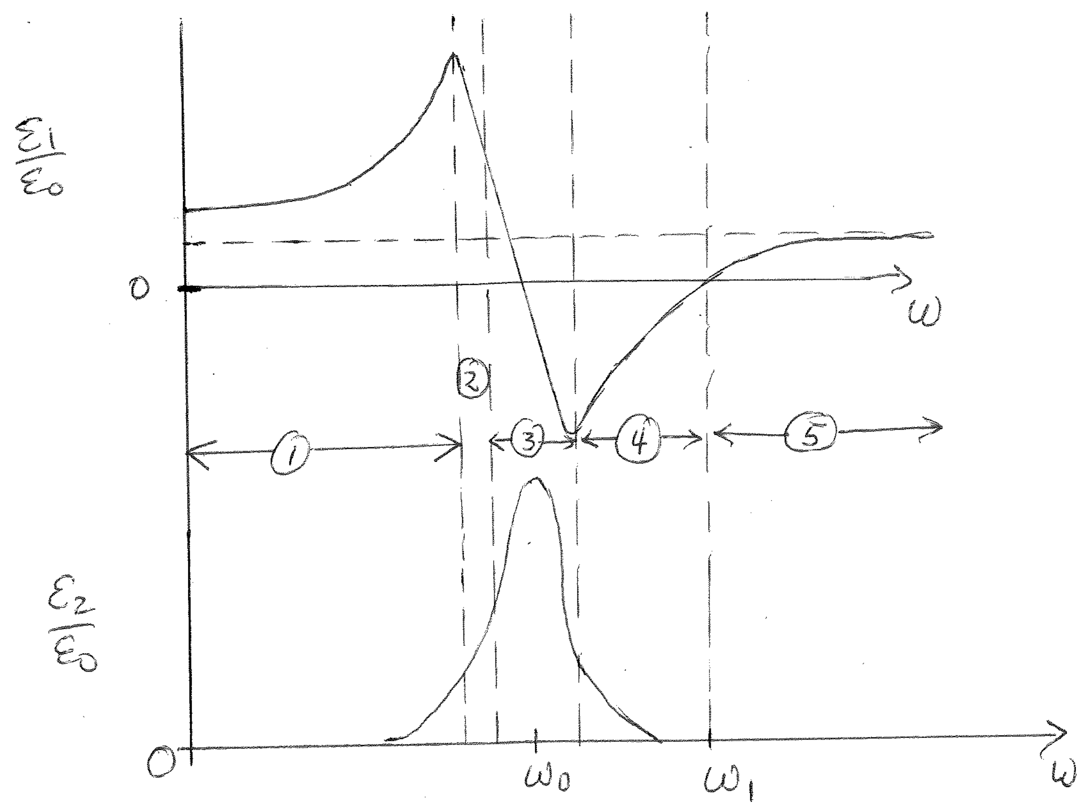
$$k^2 = k_1^2 - k_2^2 + 2 i k_1 k_2 = \frac{\omega^2}{c^2} \frac{\varepsilon_1}{\varepsilon_0} + i \frac{\omega^2}{c^2} \frac{\varepsilon_2}{\varepsilon_0}$$

equate real and imaginary pieces, and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right) \right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c} \left\{ \frac{1}{2} \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right) \right]^{1/2}$$

<u>Regions of different behavior</u>



Regions ① and ⑤: transparent propagation

$\varepsilon_1 > 0 \qquad \varepsilon_1 \gg \varepsilon_2$

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\sqrt{1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

use $\sqrt{1+x} \simeq 1 + \frac{x}{2}$ small x

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1 + \frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{\varepsilon_1}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} \simeq \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$$

$$k_2 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1 + \frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right)$$

so $k_2 \ll k_1$ small attenuation $\Rightarrow$ transparent propagation

index of refraction $n = \frac{ck_1}{\omega} \cong \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$

$\frac{dn}{d\omega} > 0 \Rightarrow$ normal dispersion

phase velocity $v_p = \frac{\omega}{k_1} = \frac{c}{n} = c\sqrt{\frac{\varepsilon_0}{\varepsilon_1}}$

in region ① $\frac{\varepsilon_1}{\varepsilon_0} > 1 \Rightarrow v_p < c$

in region ⑤ $\frac{\varepsilon_1}{\varepsilon_0} < 1 \Rightarrow v_p > c$! (but $v_g < c$ always)