Region ② : Similar to region ①, except that

$$\frac{dn}{d\omega} < 0 \Rightarrow \underline{\text{anomalous}} \text{ dispersion}$$

Region ③ : $\omega \approx \omega_0$ resonant absorption

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$

for a sharp resonance $\frac{\gamma}{\omega_0} \ll 1$
(typically $\omega_p \gg \omega_0$)

So $\varepsilon_2 \gg \varepsilon_1$

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \sqrt{1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \left(1 + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right) + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}}$$

$$k_2 \simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} - \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}}$$

$k_1 \approx k_2 \Rightarrow$ strong attenuation

wave is exciting atoms near their resonant frequency ω_0 $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave. Wave decays rapidly ~~within~~ (factor e^{-1}) within one wave length of propagation.

Region ④: $\quad \varepsilon_1 < 0 \,, \quad |\varepsilon_1| \gg \varepsilon_2 \qquad$ total reflection

* width of this region is $\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$ increases with atomic density (as $\omega_p \gg \omega_0$)

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2^2}{|\varepsilon_1| \varepsilon_0} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

cancel

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}} \frac{\varepsilon_2}{2|\varepsilon_1|}$$

$$k_2 = \pm \frac{\omega}{c} \left[\frac{1}{2} \left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2^2}{|\varepsilon_1| \varepsilon_0} - \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2} \qquad \frac{\varepsilon_1}{\varepsilon_0} = - \left| \frac{\varepsilon_1}{\varepsilon_0} \right|$$

$$= \pm \frac{\omega}{c} \left[\left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2}{|\varepsilon_1| \varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}$$

So $\quad \dfrac{k_2}{k_1} = \dfrac{2|\varepsilon_1|}{\varepsilon_2} \gg 1$

wave vector k is almost pure imaginary (since $\frac{\varepsilon_2}{\varepsilon_0} \ll 1$)
Wave decays exponentially $\to 0$ before traveling even one wavelength into material.

We will see that this is a region of total reflection. Since $\omega \gg \omega_0$, not at resonance, material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms.

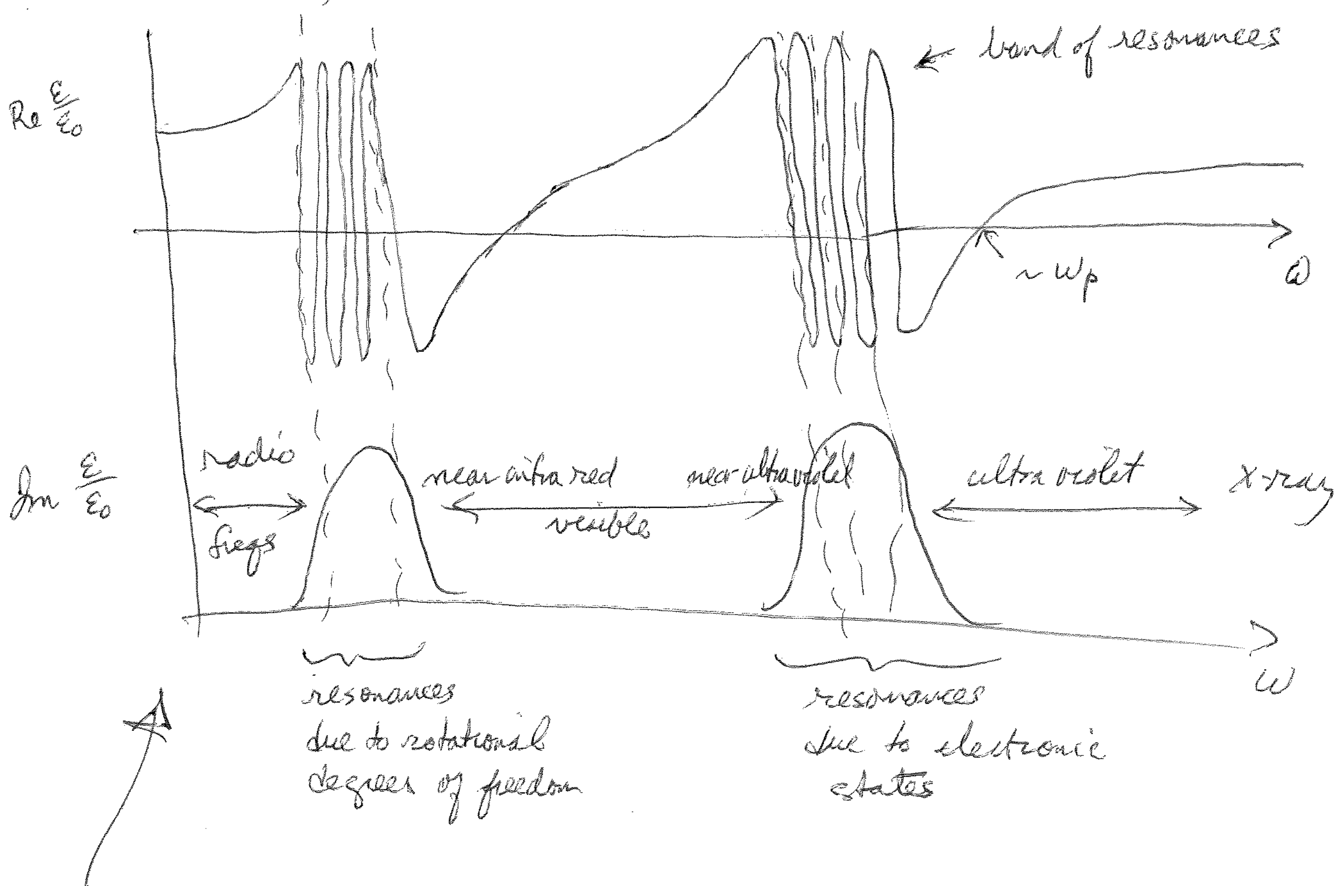
Our simple model was

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \leftarrow \text{single resonance at } \omega \approx \omega_0$$

A more realistic model of an atom or molecule would give many resonances

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are the energy spacings between quantized electron ~~ele~~ energy levels with an allowed electric dipole transition.



for a typical molecular gas

$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

$$\omega_p = c\sqrt{\frac{N_A}{\varepsilon_0}\frac{e^2}{mc^2}}\sqrt{\frac{N}{N_A}}$$

$$\frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8\times10^{-13}\ \text{cm}$$

$c = 3\times10^{10}$ cm/sec

$N_A = 6\times10^{23}\ \text{cm}^{-3}$ Avogadro's #

$$\omega_p = 4.4\times10^{16}\sqrt{\frac{N}{N_A}}\ \text{sec}^{-1}$$

$$\hbar\omega_p = 18.5\sqrt{\frac{N}{N_A}}\ \text{eV}$$

typical densities for H_2O or other $\underset{\text{liquid}}{\overset{\text{dielectric}}{}}$ liquid $\quad \frac{N}{N_A} \simeq .05$

$\hbar\omega_p \simeq 40$ eV

compared to $\hbar\omega_0 \sim$ eV

for a metal, typical densities $\quad \frac{N}{N_A} \simeq \frac{5\times10^{22}\ \text{cm}^{-3}}{6\times10^{23}\ \text{cm}^{-3}} \simeq 0.1$

$\omega_p \simeq 10^{16}\ \text{sec}^{-1}$

$\hbar\omega_p \simeq 58$ eV

Conductors

conduction electrons are free $\Rightarrow$ give $\vec{J}_f$ and ρ_f

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}} \qquad \tau \text{ is "collision time"}$$

$$\ddot{\vec{r}} + \frac{\dot{\vec{r}}}{\tau} = -\frac{e}{m}\vec{E}$$

just like polarizable atom except $\omega_0 = 0$ — no restoring force

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}$$

$$\Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t}$$

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

$$= -\frac{e\tau}{m\omega}\frac{i}{1-i\omega\tau}\vec{E}_\omega$$

current flowing is $\vec{J}_f = -eN\vec{v}$

$= -eN\dot{\vec{r}}$ $\qquad$ $N =$ density conduction electrons

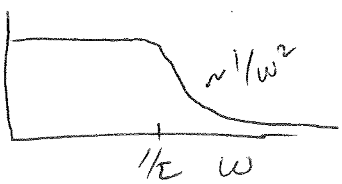
$$\vec{J}_f = \vec{J}_\omega e^{-i\omega t}, \quad \vec{J}_\omega = -eN(-i\omega)\vec{r}_\omega$$

$$= \frac{Ne^2\tau}{m}\frac{1}{1-i\omega\tau}\vec{E}_\omega$$

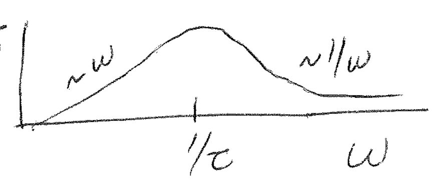
Define freq dependent conductivity

$$\boxed{\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega} \Rightarrow \boxed{\sigma(\omega) = \frac{Ne^2\tau}{m}\frac{1}{1-i\omega\tau}}$$

Re σ



Im σ



$$\text{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2}$$

$$\text{Im}\,\sigma = \frac{\sigma_0\omega\tau}{1+\omega^2\tau^2}$$

charge ~~current~~ density obtained by charge conservation

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{J}_f$$

For a plane wave $\vec{J}_f = \vec{J}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\rho_f = \rho_\omega e^{i(k\cdot r-\omega t)}$$

$$-i\omega\rho_\omega = -i\vec{k}\cdot\vec{J}_\omega \quad \Rightarrow \quad \boxed{\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega}}$$

Maxwell's Equ $\quad \vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ etc.

1) $\vec{\nabla}\cdot\vec{D} = \rho_{free}$ $\quad$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\quad$ 4) $\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_{free}$

assume $\vec{H} = \frac{\vec{B}}{\mu}$, μ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\quad$ $\varepsilon_b(\omega)$ dielectric response from bound electrons

$\vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$ $\quad$ $\sigma(\omega)$ conductivity due to free electrons

$$\rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = \frac{\sigma(\omega)}{\omega}\vec{k}\cdot\vec{E}_\omega$$