1) $\vec{\nabla}\cdot\vec{D} = \vec{\rho}_f \Rightarrow -i\vec{k}\cdot\vec{D}_\omega = \rho_\omega$

$$\Rightarrow i\vec{k}\cdot \varepsilon_b(\omega)\vec{E}_\omega = \frac{\sigma(\omega)}{\omega}\vec{k}\cdot\vec{E}_\omega$$

$$i\vec{k}\cdot\vec{E}_\omega \left[\varepsilon_b(\omega) + i\frac{\sigma(\omega)}{\omega}\right] = 0$$

2) $i\vec{k}\cdot\mu\vec{H}_\omega = 0$

3) $i\vec{k}\times\vec{E}_\omega = i\omega\vec{B}_\omega = i\omega\mu\vec{H}_\omega$

4) $i\vec{k}\times\vec{H}_\omega = -i\omega\varepsilon_b(\omega)\vec{E}_\omega + \sigma(\omega)\vec{E}_\omega$

$$= -i\omega\left[\varepsilon_b(\omega) + i\frac{\sigma(\omega)}{\omega}\right]\vec{E}_\omega$$

Equation have exactly the same form as for waves in a dielectric provided we use

$$\boxed{\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}}$$

transverse waves $\vec{E}\perp\vec{k}$

and replace μ_0 by μ.

dispersion relation for [transverse] waves is given by

$$\boxed{k^2 = \omega^2\mu\varepsilon = \frac{\omega^2}{c^2}\frac{\mu}{\mu_0}\frac{\varepsilon(\omega)}{\varepsilon_0}}$$

with $\varepsilon(\omega) = \varepsilon_b(\omega) + \frac{i\sigma(\omega)}{\omega}$

[Note: for transverse mode, $\vec{k} \perp \vec{E}_\omega$, so $\vec{k} \perp \vec{J}_\omega = \sigma(\omega)\vec{E}_\omega$
$\Rightarrow \rho_\omega = \frac{\vec{k}\cdot\vec{J}_\omega}{\omega} = 0$ no charge density oscillation!]

The main difference between wave propagation in dielectrics & conductors has to do with the contribution that the $\frac{i\sigma(\omega)}{\omega}$ term makes to the real & imaginary parts of $\varepsilon(\omega)$

For our simple model (Drude model)

$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$ where $\sigma_0 = \sigma(0) = \frac{Ne^2\tau}{m}$ is d.c. conductivity

① <u>Low frequencies $\omega \ll 1/\tau$, $\omega \ll \omega_0$</u> ω_0 is resonant freq of ε_b

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ <u>real</u>

$\sigma(\omega) \approx \sigma_0$ <u>real</u> $\sim \tau$

$$\boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} \approx \frac{\varepsilon_b(0)}{\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega}}$$

← gives large <u>imaginary</u> part to $\varepsilon(\omega)$
grows as $\frac{1}{\omega}$ as $\omega \to 0$
$\Rightarrow$ <u>strong dissipation</u>

② <u>High frequencies $\omega \gg 1/\tau$, $\omega \gg \omega_p$</u>

$\frac{\varepsilon_b(\omega)}{\varepsilon_0} \approx 1$

$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{iNe^2\tau}{m\omega\tau} = \frac{iNe^2}{m\omega}$ imaginary indep of τ

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 + \frac{i\sigma}{\varepsilon_0\omega} \approx 1 - \frac{Ne^2}{\varepsilon_0 m\omega^2} = \boxed{1 - \frac{\omega_p^2}{\omega^2} = \varepsilon(\omega)},$$

where $\omega_p = \sqrt{Ne^2/\varepsilon_0 m}$ is "plasma freq" of conduction electrons

① Behavior at low freq

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = \frac{\varepsilon_b(0)}{\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega} = \frac{\varepsilon_b(0)}{\varepsilon_0}\left(1 + \frac{i\sigma_0}{\varepsilon_b(0)\omega}\right)$$

Dissipation is due to $\varepsilon_2 = \text{Im}\,\varepsilon$

Dissipation dominate when $\varepsilon_2 \gg \varepsilon_1 = \text{Re}\,\varepsilon$

ie when $\dfrac{\sigma_0}{\varepsilon_b(0)\omega} \gg 1$ — this regime is called a "good" conductor – conduction electrons playing dominant role — waves strongly attenuated

opposite limit : $\dfrac{\sigma_0}{\varepsilon_b(0)\omega} \ll 1$ — this regime is called a "poor" conductor – waves propagate transparently – little relative absorption of energy from conduction electrons

One always gets into the "good" conductor limit as ω decreases. For good conductor,

$$k = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}} \simeq \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}}\sqrt{i\frac{\varepsilon_2}{\varepsilon_0}} = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}}\sqrt{\frac{\sigma_0}{\varepsilon_0\omega}}\sqrt{i}$$

$$k = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\frac{\sigma_0}{\varepsilon_0\omega}}\left(\frac{1+i}{\sqrt{2}}\right) \Rightarrow k_1 = k_2$$

real and imaginary parts of k are equal

$\frac{1}{c} = \sqrt{\mu_0 \varepsilon_0}$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{\mu}{\varepsilon_0 \mu_0}\frac{\sigma_0}{2\omega}} = \sqrt{\frac{\mu \sigma_0}{2}\omega} \sim \sqrt{\omega}$$

waves have form $\vec{E} = E_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$

decay length of amplitude is

$$1/k_2 = \sqrt{\frac{2}{\mu \sigma_0 \omega}} = \delta \text{ called the "skin depth"}$$

Faraday cage

δ is distance wave penetrates into conductor

$\delta \sim 1/\sqrt{\omega}$ gets larger as ω decreases

$$\vec{H} = \vec{H}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t + \phi)} \qquad \left|\frac{\vec{H}_\omega}{E_\omega}\right| = \frac{|k|}{\omega\mu}$$

phase shift between $\vec{H}$ and $\vec{E}$ is ϕ

given by $\tan\phi = k_2/k_1 \approx 1$

$$\Rightarrow \phi \approx 45^\circ$$

Amplitude ratio $\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{|k|}{\omega\mu} = \frac{\sqrt{2}\, k_1}{\omega\mu} = \frac{\sqrt{2}}{\omega\mu}\sqrt{\frac{\mu\sigma_0\omega}{2}}$

$= \sqrt{\frac{\sigma_0}{\omega\mu}}$ increases as $\frac{1}{\sqrt{\omega}}$ as $\omega \to 0$

$\Rightarrow$ as $\omega \to 0$, most of energy of wave is carried by the magnetic field part of the wave.

② Behavior at high freq

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \left(\frac{\omega_p}{\omega}\right)^2 \qquad \omega_p^2 = \frac{Ne^2}{m\varepsilon_0}$$ plasma freq

$\varepsilon(\omega)$ is real $(\varepsilon_2 \ll \varepsilon_1)$

1) If $\omega > \omega_p$, then $\varepsilon > 0$

$\Rightarrow$ transparent propagation $\quad k_1 = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

k is pure real $\quad k_2 \approx 0$

2) If $\omega < \omega_p$, then $\varepsilon < 0$

$\Rightarrow$ total reflection $\quad k_1 = 0$

$$k_2 = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\left(\frac{\omega_p^2}{\omega^2} - 1\right)}$$

k is pure imaginary

plasma freq ω_p gives cross over between reflection + transparent propagation.

$\tau \sim 10^{-14}$ sec for typical metal

$\omega_p \simeq 10^{16}\ \text{sec}^{-1}$ for most metals

$$\lambda_p \equiv \frac{2\pi c}{\omega_p} \sim 3\times10^3\ \text{Å}$$ (visible is $\lambda \sim 5\times10^3$ Å)

Example: The ionosphere is a layer of charged gas surrounding the earth. In may respects the charged gas behaves like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio propagates through ionosphere + escapes into space

Explains why you can pick up AM stations from far away - they are reflected back by ionosphere - but you only pick up local FM stations - they do not get reflected by ionosphere.

What about longitudinal modes? (ie H_ω, E_ω not $\perp \vec{k}$)

magnetic field

$i\mu \vec{k} \cdot \vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k}$ or $\vec{k} = 0$ uniform magnetic field

Faraday $\quad i\vec{k} \times \vec{E}_\omega = i\omega\mu \vec{H}_\omega \Rightarrow \omega = 0$

$\quad\quad \overset{\parallel}{0}$ as $\vec{k} = 0$

($\vec{H} \perp \vec{k}$ would be transverse mode so longitudinal mode must have $k=0$ and so $\omega = 0$.

so only possible longitudinal magnetic field is a spatially uniform, constant in time $\vec{H}$.

electric field

$i\,\varepsilon(\omega)\vec{k} \cdot \vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k}$, or $\vec{k} = 0$, or $\varepsilon(\omega) = 0$!

we can satisfy all Maxwell's equations for a $\vec{E}_\omega \parallel \vec{k}$, provided $\varepsilon(\omega) = 0$, and by above, $\vec{H}_\omega = 0$ for this mode.

$i\vec{k} \times \vec{E}_\omega = i\omega\mu \vec{H}_\omega$ — both sides vanish.

LHS $= 0$ as $\vec{E}_\omega \parallel \vec{k} \Rightarrow \vec{k} \times \vec{E}_\omega = 0$

RHS $= 0$ as $\vec{H}_\omega = 0$

$i\vec{k} \times \vec{H}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$ — LHS $= 0$ as $\vec{H}_\omega = 0$

RHS $= 0$ as $\varepsilon(\omega) = 0$

$i\mu \vec{k} \cdot \vec{H}_\omega = 0$ — satisfied as $\vec{H}_\omega = 0$

So we can have a longitudinal ~~oscillation~~ $\vec{E}$ provided $\varepsilon(\omega) = 0$

frequencies of longitudinal mode given by $\varepsilon(\omega) = 0$.

<u>low freq</u> $\omega \ll \omega_0$, $\omega\tau \ll 1$

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_b}{\varepsilon_0} + \frac{i\sigma}{\varepsilon_0\omega} \simeq 1 + \frac{Na\,e^2}{m\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0\omega} = \frac{1}{\varepsilon_0}\left(\varepsilon_b(0) + \frac{i\sigma_0}{\omega}\right)$$

$$\frac{\varepsilon}{\varepsilon_0} = 0 \quad \text{when} \quad \omega = \frac{-i\sigma_0}{\varepsilon_b(0)}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{-\sigma_0 t/\varepsilon_b(0)} e^{i\vec{k}\cdot\vec{r}}$$

$\Rightarrow$ if set up a longitudinal $\vec{E}$ field, it decays to zero exponentially fast, with decay time $\frac{\varepsilon_b(0)}{\sigma_0}$

Consistant with our assumption that $\vec{E} = 0$ inside a conductor for electrostatics.

(electrostatic fields are <u>always</u> longitudinal

$$\vec{E} = -\vec{\nabla}V \Rightarrow \cancel{\vec{E}_k = i\vec{k}V_k \text{ for Fourier component}}$$

$$\vec{E} \sim -i\vec{k}V_k e^{i\vec{k}\cdot\vec{r}} \qquad \vec{E} \sim \vec{k}$$