planes of constant <u>phase</u> are $\perp$ to $\vec{k}_{T1}$

Now we solve for k_{T1} and k_{T2} and θ_{T1}

Dispersion relation in medium 2: $k_T^2 = \omega^2 \mu_b \varepsilon_b$

$$k_T^2 = (\vec{k}_{T1} + i\vec{k}_{T2})^2 = k_{T1}^2 - k_{T2}^2 + 2i\vec{k}_{T1}\cdot\vec{k}_{T2}$$
$$= k_{T1}^2 - k_{T2}^2 + 2i k_{T1} k_{T2} \cos\theta_{T1} \qquad (\text{since } \theta_{T2} = 0)$$
$$= \omega^2 \mu_b (\varepsilon_{b1} + i\varepsilon_{b2})$$

equate real and imaginary parts of both sides

$$\left.\begin{aligned} k_{T1}^2 - k_{T2}^2 &= \omega^2 \mu_b \varepsilon_{b1} \\ 2k_{T1}k_{T2} &= \frac{\omega^2 \mu_b \varepsilon_{b2}}{\cos\theta_{T1}} \end{aligned}\right\}$$

same equations as when we considered propagatn in an infinite dielectric, only then $\theta_{T1} = 0$

Consider the above as two equations for two unknowns k_{T1} and k_{T2}. Solve for k_{T1} and k_{T2} in terms of $\cos\theta_{T1}$

$$k_{T1} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2}$$

$$k_{T2} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} - \frac{\varepsilon_{b1}}{2}\right]^{2}$$

If $\theta_{T1} = 0$, this is the same as our earlier result

Finally we use our boundary condition to determine θ_{T1}

$$k_{T1}\sin\theta_{T1} = k_I \sin\theta_I$$

$$k_I = \omega\sqrt{\mu_a \varepsilon_a} = \frac{\omega}{c}\sqrt{\frac{\mu_a\varepsilon_a}{\mu_0\varepsilon_0}} = \frac{\omega}{c} n_a$$

← index of refraction

$$k_{T1} = \frac{k_I \sin\theta_I}{\sin\theta_{T1}} = \frac{\omega}{c}\, \frac{n_a \sin\theta_I}{\sin\theta_{T1}}$$

$$= \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2}$$

$$\Rightarrow \boxed{n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2} \sin\theta_{T1}}$$

↑ determines angle of transmission θ_{T1} in terms of angle of incidence θ_I and the physical parameters n_a, μ_b, ε_{b1}, ε_{b2} of the two materials

Cases ① If material b is transparent, ie $\varepsilon_{b2} \ll \varepsilon_{b1}$

define $n_b = \sqrt{\frac{\mu_b \varepsilon_{b1}}{\mu_0 \varepsilon_0}} = \sqrt{\mu_b \varepsilon_{b1}}\, c$

then $n_a \sin\theta_I = n_b \sin\theta_{T1}\left[\frac{1}{2\varepsilon_{b1}}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$

$$= n_b \sin\theta_{T1}\left[\frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$$

expand the $\sqrt{1+\delta} \approx 1 + \delta/2$

$$\approx n_b \sin\theta_{T1}\left[\frac{1}{2} + \frac{\varepsilon_{b2}^2}{4\varepsilon_{b1}^2\cos^2\theta_{T1}} + \frac{1}{2}\right]^{1/2}$$

$$= n_b \sin\theta_{T1}\left[1 + \frac{\varepsilon_{b2}^2}{4\varepsilon_{b1}^2\cos^2\theta_{T1}}\right]^{1/2}$$

expand the $\sqrt{\ }$

$$n_a \sin\theta_I \simeq n_b \sin\theta_{T1}\left[1 + \underbrace{\frac{\varepsilon_{b2}^2}{8\varepsilon_{b1}^2\cos^2\theta_{T1}}}_{\text{small correction to Snell's Law}}\right]$$

when $\frac{\varepsilon_{b2}}{\varepsilon_{b1}} \ll 1$, we can solve above equation interatively to get approximate result

$$n_a \sin\theta_I = n_b \sin\theta_{T1}\left[1 + \text{small}\right]$$

$$\Rightarrow \sin\theta_{T1} \approx \frac{n_a}{n_b}\sin\theta_I \quad \Rightarrow \cos^2\theta_{T1} \approx 1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I$$

so to next order

$$n_b \sin\theta_{T1} \simeq \frac{n_a \sin\theta_I}{1 + \frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2\left[\frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}\right]}$$

$$\simeq n_a \sin\theta_I\left[1 - \underbrace{\frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2 \frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}}_{\text{this term is } > 0 \text{ so} \dots}\right]$$

$$\lesssim n_a \sin\theta_I$$

Result is that θ_{T1} is smaller than one would predict from Snell's law.

The correction is of order $O\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2$.

medium b is a

Case (2) good conductor or a region of resonant absorption of a dielectric so $\varepsilon_{b2} \gg \varepsilon_{b1}$

Now, to lowest order we will approx $\varepsilon_{b1} \approx 0$ then

$$n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\frac{\varepsilon_{b2}}{\cos\theta_{T1}}\right]^{1/2} \sin\theta_{T1}$$

$$\boxed{n_a \sin\theta_I = c\sqrt{\frac{\mu_b \varepsilon_{b2}}{2}}\,\frac{\sin\theta_{T1}}{\sqrt{\cos\theta_{T1}}}}$$

determines θ_{T1} in terms of θ_I

In this case our result for θ_{T1} looks nothing like Snell's Law.

$\Rightarrow$ Snell's Law only holds if both the media at transparent at the frequency of interest

So far, all our results come from the requirement that the phases of the incident, reflected, and transmitted waves all match at the interface. This is enough to determine the directions, wavelengths, attenuation, and frequencies of the waves. These results hold for <u>any</u> type of wave, not just electromagnetic waves.

Now want to solve for amplitudes of transmitted and reflected waves.

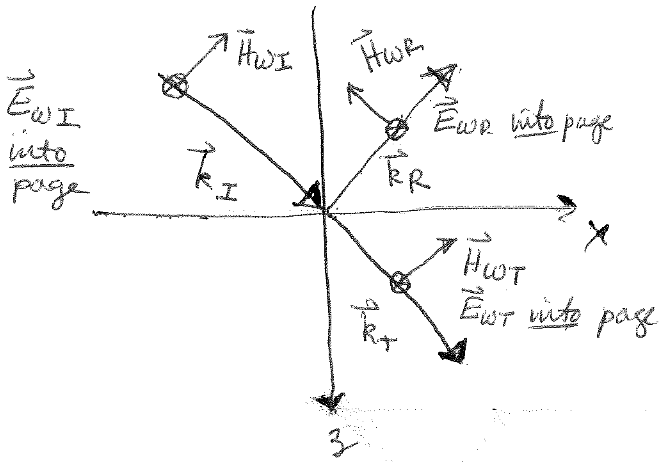
<u>two cases:</u> "plane of incidence" = plane spanned by the wavevector $\vec{k}_I$, and the normal to the interface — in our case, the xz plane

① $\vec{E}_\omega$ is $\perp$ to the plane of incidence

② $\vec{E}_\omega$ is $\parallel$ to the plane of incidence

the most general case is a linear superposition of these two, so treating these two cases separately also gives the general solution.

$E_0 \perp$ plane of incidence



($H_{\omega I}$ in plane of incidence)

all the $\vec{E}$'s are along $\hat{y}$

Continuity of $\hat{y}$ components of $\vec{E}$

(1) $\quad E_I + E_R = E_T$

where $\vec{E}_{\omega I} = E_I \hat{y}$ etc.

continuity of $\hat{x}$ components of $\vec{H}$

$$H_{Ix} + H_{Rx} = H_{Tx}$$

Faraday $\quad H_x = \dfrac{k_z}{\omega\mu} E_y$

$\mu\omega \vec{H} = \vec{k} \times \vec{E}$

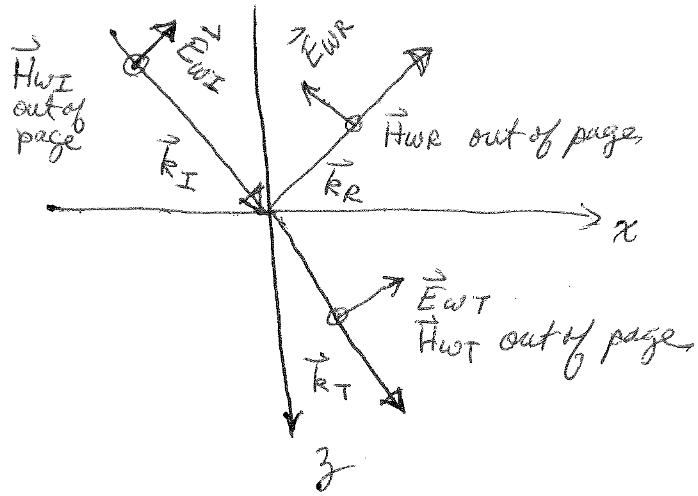
plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

(2) $$\frac{k_{Iz}}{\mu_a}(E_I - E_R) = \frac{k_{Tz}}{\mu_b} E_T$$

solve equation (1) and (2) for E_R and E_T in terms of E_I

$E_0 \parallel$ plane of incidence



($H_{\omega I} \perp$ to plane of incidence)

all the $\vec{H}$'s are along $\hat{y}$

Continuity of $\hat{y}$ components of $\vec{H}$

(1) $\quad H_I + H_R = H_T$

where $\vec{H}_{\omega I} = H_I \hat{y}$ etc.

continuity of $\hat{x}$ components of $\vec{E}$

$$E_{Ix} + E_{Rx} = E_{Tx}$$

Ampere $\quad E_x = \dfrac{-k_z}{\omega\varepsilon} H_y$

$-i\omega\varepsilon \vec{E} = i\vec{k} \times \vec{H}$

plug in above and use $k_{Iz} = -k_{Rz}$

$\Rightarrow$

(2) $$\frac{k_{Iz}}{\varepsilon_a}(H_I - H_R) = \frac{k_{Tz}}{\varepsilon_b} H_T$$

solve equations (1) and (2) for H_R and H_T in terms of H_I

$$E_R = \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} E_I \qquad H_R = \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} H_I$$

$$E_T = \frac{2\mu_b k_{Iz}}{\mu_a k_{Tz} + \mu_b k_{Iz}} E_I \qquad H_T = \frac{2\varepsilon_b k_{Iz}}{\varepsilon_a k_{Tz} + \varepsilon_b k_{Iz}} H_I$$

We can now define the reflection and transmission coefficients. These are defined in terms of the transported energy. Since the energy flux $|\vec{S}|$ is $\sim |\vec{E}|^2 \sim |\vec{H}|^2$, we have

Reflection coefficient

① $E_0 \perp$ to plane of incidence

$$R_\perp = \frac{|E_R|^2}{|E_I|^2} = \left| \frac{\mu_b k_{Iz} - \mu_a k_{Tz}}{\mu_b k_{Iz} + \mu_a k_{Tz}} \right|^2$$

② $E_0 \parallel$ to plane of incidence

$$R_\parallel = \frac{|H_R|^2}{|H_I|^2} = \left| \frac{\varepsilon_b k_{Iz} - \varepsilon_a k_{Tz}}{\varepsilon_b k_{Iz} + \varepsilon_a k_{Tz}} \right|^2$$

For region of "total reflection" in material b, $\mathrm{Im}\,\varepsilon_b \simeq 0$, $\mathrm{Re}\,\varepsilon_b < 0$

$\Rightarrow \vec{k}_T \equiv i \vec{K}_T$ where $\vec{K}_T$ is real ($\vec{k}_T$ is pure imaginary)

$$\Rightarrow R_\perp = \left| \frac{\mu_b k_{Iz} - i\mu_a K_{Tz}}{\mu_b k_{Iz} + i\mu_a K_{Tz}} \right|^2$$

$$R_\parallel = \left| \frac{\varepsilon_b k_{Iz} - i\varepsilon_a K_{Tz}}{\varepsilon_b k_{Iz} + i\varepsilon_a K_{Tz}} \right|^2$$

both are of the form

$$\left| \frac{a - ib}{a + ib} \right|^2 = 1$$

when a, b both real

Additional notes on Reflection & Transmission Coefficients

For a transparent medium, the energy current can be written as (see text 9-3.1)

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \vec{E}\times\vec{H} \qquad (\text{in a vacuum } \mu = \mu_0)$$

For a plane wave $\vec{E}(\vec{r},t) = \vec{E}_\omega \cos(\vec{k}\cdot\vec{r}-\omega t) \qquad \vec{E}_\omega \perp \vec{k}$

From lecture 13 we have

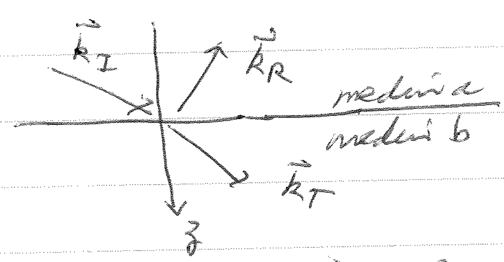
$$\vec{H}(\vec{r},t) = \frac{\vec{B}(\vec{r},t)}{\mu} = (\hat{k}\times\vec{E}_\omega)\frac{|k|}{\omega\mu}\cos(\vec{k}\cdot\vec{r}-\omega t)$$

(since the medium is transparent, k is real and $k_2 = 0$, $\delta = \arctan\frac{k_2}{k_1} = 0$)

$$\Rightarrow \quad \vec{S} = \vec{E}\times\vec{H} = \frac{|k|}{\omega\mu}\underbrace{\vec{E}_\omega\times(\hat{k}\times\vec{E}_\omega)}_{=|\vec{E}_\omega|^2\hat{k} \text{ since } \hat{k}\perp\vec{E}_\omega}\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

So

$$\langle\vec{S}\rangle = \frac{|k|}{2\omega\mu}|\vec{E}_\omega|^2\hat{k} \qquad \text{as } \langle\cos^2(\vec{k}\cdot\vec{r}-\omega t)\rangle = 1/2$$



The energy flux of the incident wave going into medium b is

$$\langle\vec{S}_I\rangle\cdot\hat{z} = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2(\hat{k}_I\cdot\hat{z}) = \frac{|k_I|}{2\omega\mu_a}|\vec{E}_{I\omega}|^2\cos\theta_I$$

The energy flux of the reflected wave is

use $\theta_I = \theta_R$

$$\langle\vec{S}_R\rangle\cdot\hat{z} = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(\hat{k}_R\cdot\hat{z}) = \frac{|k_R|}{2\omega\mu_a}|\vec{E}_{R\omega}|^2(-\cos\theta_I)$$

(–) because reflected back

The energy flux of the transmitted wave is

$$\langle\vec{S}_T\rangle\cdot\hat{z} = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2(\hat{k}_T\cdot\hat{z}) = \frac{|k_T|}{2\omega\mu_b}|\vec{E}_{T\omega}|^2\cos\theta_T$$

Energy in = Energy out

$$\Rightarrow \langle \vec{S}_I \rangle \cdot \hat{z} + \langle \vec{S}_R \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z}$$

or $\langle \vec{S}_I \rangle \cdot \hat{z} = \langle \vec{S}_T \rangle \cdot \hat{z} - \langle \vec{S}_R \rangle \cdot \hat{z}$

this term is >0 this term is <0

$$|\langle \vec{S}_I \rangle \cdot \hat{z}| = |\langle \vec{S}_T \rangle \cdot \hat{z}| + |\langle \vec{S}_R \rangle \cdot \hat{z}|$$

If we define $R = \dfrac{|\langle \vec{S}_R \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$, $T = \dfrac{|\langle \vec{S}_T \rangle \cdot \hat{z}|}{|\langle \vec{S}_I \rangle \cdot \hat{z}|}$

then we get

$$1 = T + R$$

Also

$$R = \frac{\frac{|k_R|}{2\omega\mu_a} |\vec{E}_{RW}|^2 \cos\theta_I}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{IW}|^2 \cos\theta_I} = \frac{|\vec{E}_{RW}|^2}{|\vec{E}_{IW}|^2} \quad \text{since } |k_I| = |k_R|$$

But

$$T = \frac{\frac{|k_T|}{2\omega\mu_b} |\vec{E}_{TW}|^2 \cos\theta_T}{\frac{|k_I|}{2\omega\mu_a} |\vec{E}_{IW}|^2 \cos\theta_I} \neq \frac{|\vec{E}_{TW}|^2}{|\vec{E}_{IW}|^2}$$