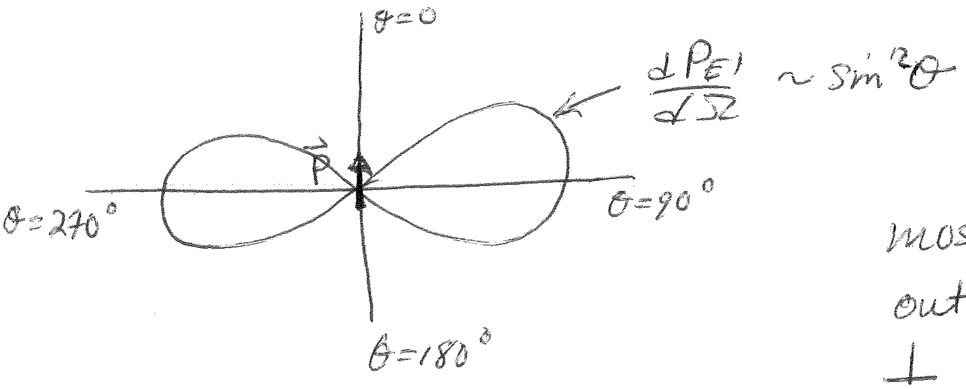
$$\frac{dP_{E1}}{d\Omega} = \hat{r}\cdot\langle\vec{S}_{E1}\rangle r^2 = \frac{ck^4p^2}{2(4\pi)^2\varepsilon_0}\sin^2\theta \sim \omega^4\sin^2\theta$$



most of power is directed outwards into the plane $\perp$ to $\vec{p}$, ie at angles θ peaked about 90°

For energy conservation to hold, it must be true that all the higher order terms, that go as higher powers of $\frac{1}{r}$ (ie $\frac{1}{r^2}$, $\frac{1}{r^3}$, etc ...), must vanish when compute the time averaged energy flux integrated over surface of sphere otherwise energy would be dissappearing as the wave propagated outwards.

Total power radiated is

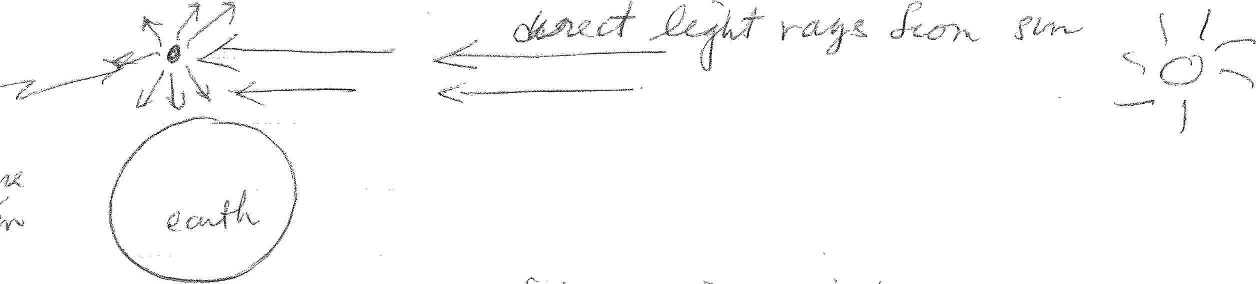
$$P_{E1} = \int \frac{dP_{E1}}{d\Omega}\,d\Omega = \frac{ck^4p^2}{2(4\pi)^2\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\,\sin^2\theta$$

$$= \frac{ck^4p^2}{32\pi^2\varepsilon_0}\,2\pi\,\underbrace{\int_0^{\pi}d\theta\,\sin\theta(1-\cos^2\theta)}_{\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{4}{3}}$$

$$P_{E1} = \frac{ck^4p^2}{4\pi\varepsilon_0\cdot 3} = \boxed{\frac{p^2\omega^4}{4\pi\varepsilon_0\;3c^3} = P_{E1} \sim \omega^4}$$

Why the sky is blue - Lord Rayleigh

When look at sky, you are seeing the indirect light of the sun, which is the light emitted by the atoms and ~~molect~~ molecules of the atmosphere as they oscilate & so radiate, due to the electric field of the direct light from the sun



molecule, ~~atom~~ in atmosphere oscilates in $\vec{E}$ field of direct rays, and then emits radiated light with power (can view this as a scattering of the direct rays)

$$P \sim \omega^4 p^2$$

polarizability

p is dipole moment of ~~atom~~ molecule in atmosphere $p = \alpha E$

electric field of direct rays

$$\alpha \sim \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

for molecules in atmosphere, N_2, etc, ω_0 is typically a freq higher than the visible spectrum. Therefore for light in visible spectrum, $\alpha \sim \frac{e^2}{m\omega_0}$ indep of ω

Power emitted $\sim \omega^4$ largest at higher freq.

Since light from sun is "white light" it has components of all freqs. ~~For~~ From the above, we see that this ~~indirect scattered~~ light is most scattered at the higher freqs, due to the ω^4 dependence of

scattered power in electric dipole approx
⇒ indirect light is strongest in the blue (large ω)
part of the visible spectrum. ⇒ sky is blue!

When we look at sunrise or sunset however,
we are looking at the direct rays of the sun.
Since these rays are scattered most in the blue,
the direct rays are strongest in the red (small ω)
part of the spectrum ⇒ sunset & sunrise are red!

Note: when we wrote

$$Re\left[\vec{E}_{E1}(\vec{r},\omega)e^{-i\omega t}\right] = \frac{k^2}{4\pi\epsilon_0}\frac{\cos(kr-\omega t)}{r}\hat{r}\times(\vec{p}\times\hat{r})$$

we implicitly assumed that the amplitude of the oscillating electric dipole moment $\vec{p}(\omega)$ was a real vector. But that is not necessarily always the case!

For $\vec{p}(\omega) \equiv \vec{p}_1$ real, the time dependent dipole moment is

$$\vec{p}(t) = Re\left[\vec{p}_1 e^{-i\omega t}\right] = \vec{p}_1 \cos\omega t$$

points always in same direction with oscillating magnitude.

But suppose $\vec{p}(\omega) = \vec{p}_1 + i\vec{p}_2$. Then

$$\vec{p}(t) = Re\left[(\vec{p}_1+i\vec{p}_2)e^{-i\omega t}\right]$$
$$= \vec{p}_1\cos\omega t + \vec{p}_2\sin\omega t$$

Now direction of $\vec{p}(t)$ is rotating!

if $\vec{p}_1 \perp \vec{p}_2$ then the tip of $\vec{p}(t)$ sweeps out an ellipse!

So if $\vec{p}(\omega)$ is complex we need to be more careful in our calculation of $\vec{S}$

Magnetic Dipole Radiation - in the Radiation Zone Approx

$$\vec{A}_{M1} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, ik\,\hat{r}\times\vec{m}$$

$$\vec{B}_{M1} = \vec{\nabla}\times\vec{A}_{M1} = \frac{\mu_0}{4\pi}\, ik\,\vec{\nabla}\times\left(e^{ikr}\,\frac{\hat{r}\times\vec{m}}{r}\right)$$

Exactly the same form as when we computed $\vec{E}_{E1}$ from B_{E1} except $\vec{p}\rightarrow\vec{m}$

use $\vec{\nabla}\times(f\vec{g}) = \vec{\nabla}f\times\vec{g} + f\,\vec{\nabla}\times\vec{g}$

$$\vec{B}_{M1} = \frac{\mu_0}{4\pi}\, ik\left[\underbrace{(\vec{\nabla}e^{ikr})\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)}_{ike^{ikr}\,\hat{r}\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)} + \underbrace{e^{ikr}\,\vec{\nabla}\times\left(\frac{\hat{r}\times\vec{m}}{r}\right)}_{\sim O\left(\frac{1}{r^2}\right)\text{ so ignore in RZ approx}}\right]$$

$$\vec{B}_{M1} = \frac{-\mu_0}{4\pi}\, k^2\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times\vec{m})$$

$$\vec{E}_{M1} = \frac{i}{\omega\mu_0\varepsilon_0}\,\vec{\nabla}\times\vec{B}_{M1}$$ from Amperes law with $\vec{J}=0$

$$= \frac{-i\mu_0 k^2}{4\pi\omega\mu_0\varepsilon_0}\,\vec{\nabla}\times\left(\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times\vec{m})\right)$$

$$= \frac{-ik^2}{4\pi\omega\varepsilon_0}\left[\underbrace{(\vec{\nabla}e^{ikr})}_{ik\hat{r}e^{ikr}}\times\left(\frac{\hat{r}\times(\hat{r}\times\vec{m})}{r}\right) + \underbrace{e^{ikr}\,\vec{\nabla}\times\left(\frac{\hat{r}\times(\hat{r}\times\vec{m})}{r}\right)}_{\sim O\left(\frac{1}{r^2}\right)\text{ so ignore in RZ approx}}\right]$$

$$= \frac{k^3}{4\pi\omega\varepsilon_0}\,\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times(\hat{r}\times\vec{m}))$$ use $\omega = ck$

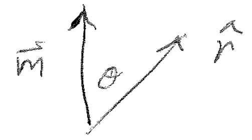
use triple product rule

$$\hat{r}\times(\hat{r}\times(\hat{r}\times\vec{m})) = \hat{r}(\hat{r}\cdot(\hat{r}\times\vec{m})) - (\hat{r}\times\vec{m})(\hat{r}\cdot\hat{r})$$
$$= 0 - \hat{r}\times\vec{m}$$

$$\vec{E}_{M1} = \frac{-k^2}{4\pi\varepsilon_0 c}\,\frac{e^{ikr}}{r}\,\hat{r}\times\vec{m}$$

$$E_{M1} = \frac{k^2}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}\left[-\hat{r}\times\vec{m}\right]$$

$$\boxed{\begin{aligned}\vec{E}_{M1} &= \frac{k^2}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}\left[\vec{m}\times\hat{r}\right]\\ \vec{B}_{M1} &= \frac{\mu_0 k^2}{4\pi}\frac{e^{ikr}}{r}\left[\hat{r}\times(\vec{m}\times\hat{r})\right]\end{aligned}}$$



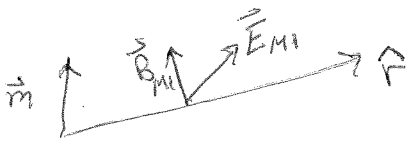
for $\underline{\underline{\vec{m} = m\hat{z}}}$, $\vec{m}\times\hat{r} = m\sin\theta\,\hat{\phi}$

$\hat{r}\times(\vec{m}\times\hat{r}) = m\sin\theta\,(-\hat{\theta})$

$$\left.\begin{aligned}\vec{E}_{M1} &= \frac{k^2 m}{4\pi\varepsilon_0 c}\frac{e^{ikr}}{r}\sin\theta\;\hat{\phi}\\ \vec{B}_{M1} &= \frac{-\mu_0 k^2 m}{4\pi}\frac{e^{ikr}}{r}\sin\theta\;\hat{\theta}\end{aligned}\right\}$$

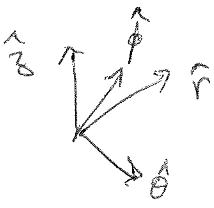
Note similarity with $\vec{E}_{E1}$ ad $\vec{B}_{E1}$

$\vec{p}\to\frac{\vec{m}}{c}$, $\vec{E}$ + $\vec{B}$ rotated by 90°



Poynting vector

~~$\vec{S}_{M1} = \frac{1}{\mu_0}\vec{E}_{M1}(\vec{r},t)\times\vec{B}_{M1}(\vec{r},t) = \frac{1}{\mu_0}k^2m$~~



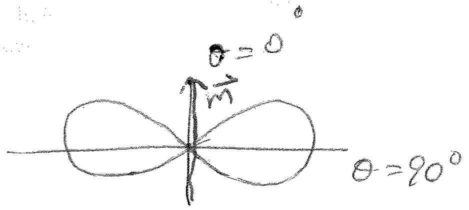
$$\vec{S}_{M1} = \frac{1}{\mu_0}\,\mathrm{Re}\left\{\vec{E}_{M1}e^{-i\omega t}\right\}\times\mathrm{Re}\left\{\vec{B}_{M1}e^{-i\omega t}\right\}$$

$$= \frac{1}{\mu_0}\left(\frac{k^2 m}{4\pi\varepsilon_0 c}\right)\left(\frac{-\mu_0 k^2 m}{4\pi}\right)\frac{\sin^2\theta}{r^2}\cos^2(kr-\omega t)\;\underbrace{\hat{\phi}\times\hat{\theta}}_{-\hat{r}}$$

$$= \frac{k^4 m^2}{4\pi\varepsilon_0\;4\pi c}\frac{\sin^2\theta}{r^2}\cos^2(kr-\omega t)\;\hat{r}$$

time average

$$\langle\vec{S}_{M1}\rangle = \frac{k^4 m^2}{32\pi^2\varepsilon_0 c}\frac{\sin^2\theta}{r^2}\;\hat{r}$$



power cross section

$$\frac{dP_{M1}}{d\Omega} = \hat{r}\cdot\langle \vec{S}_{M1}\rangle r^2 = \boxed{\frac{k^4 m^2}{2(4\pi)^2 \varepsilon_0 c}\sin^2\theta = \frac{dP_{M1}}{d\Omega}}$$

same form as $\frac{dP_{E1}}{d\Omega}$ with $p \rightarrow \frac{m}{c}$

total power

$$P_{M1} = \int \frac{dP_{M1}}{d\Omega}\, d\Omega = \frac{2\pi\, k^4 m^2}{2(4\pi)^2 \varepsilon_0 c} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3}$$

$k^4 = \frac{\omega^4}{c^4}$

$$\boxed{P_{M1} = \frac{\omega^4 m^2}{4\pi\varepsilon_0\, 3c^5}}$$

compare to $P_{E1} = \frac{\omega^4 p^2}{4\pi\varepsilon_0\, 3c^3}$

$$\frac{P_{M1}}{P_{E1}} = \left(\frac{m}{cp}\right)^2 \quad \text{as } m \sim vp \Rightarrow \frac{P_{M1}}{P_{E1}} \sim \left(\frac{v}{c}\right)^2$$

electric quadrapole radiation - radiation zone approx

$$\vec{A}_{E2} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, ik\left(\frac{i\omega}{6}\right)\hat{r}\cdot\overleftrightarrow{Q}$$

check

Find fields for homework!

For arbitrary charge distributions - not single frequency

We found that for pure freq of oscilation, with electric dipole moment

$$\vec{p}(t) = \vec{p}(\omega)\, e^{-i\omega t}$$

the radiated fields in electric dipole approx are

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},\omega)\, e^{-i\omega t} \qquad , \qquad \vec{B}(\vec{r},t) = \vec{B}(\vec{r},\omega)\, e^{-i\omega t}$$

with

$$\vec{E}(\vec{r},\omega) = \frac{k^2}{4\pi\varepsilon_0} \frac{e^{ikr}}{r}\, \hat{r} \times (\vec{p}(\omega) \times \hat{r})$$

$$= \frac{\mu_0 \omega^2}{4\pi} \frac{e^{i\omega r/c}}{r}\, \hat{r} \times (\vec{p}(\omega) \times \hat{r}) \qquad \text{using } k = \omega/c, \quad c^2 = 1/\mu_0\varepsilon_0$$

$$\vec{B}(\vec{r},\omega) = -\frac{c\mu_0 k^2}{4\pi} \frac{e^{ikr}}{r}\, \vec{p}(\omega) \times \hat{r}$$

$$= -\frac{\mu_0 \omega^2}{4\pi c} \frac{e^{i\omega r/c}}{r}\, \vec{p}(\omega) \times \hat{r}$$

For an arbitrary time varying charge, with electric dipole moment

$$\vec{p}(t) = \int d\omega\, \vec{p}(\omega)\, e^{-i\omega t}$$

Solutions are obtained by linear superposition

$$\vec{E}(\hat{r},t) = \int d\omega\, \vec{E}(\hat{r},\omega)\, e^{-i\omega t}$$

$$= \frac{\mu_0}{4\pi r}\, \hat{r} \times \left[\int d\omega\, e^{-i\omega(t - r/c)}\, \omega^2\, \vec{p}(\omega) \times \hat{r} \right]$$

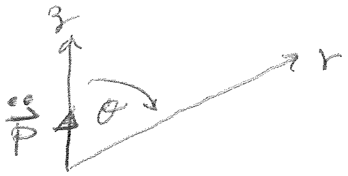
$$= \frac{\mu_0}{4\pi r}\, \hat{r} \times \left[-\frac{\partial^2}{\partial t^2} \overbrace{\int d\omega\, e^{-i\omega(t - r/c)}\, \vec{p}(\omega)}^{\vec{p}(t - r/c)} \times \hat{r} \right]$$

use triple product rule

$$\vec{E}(\vec{r},t) = \frac{-\mu_0}{4\pi r}\,\hat{r}\times\left(\ddot{\vec{p}}(t-r/c)\times\hat{r}\right)$$

second time derivative of $\vec{p}(t)$ evaluated at $\boxed{t_0 = t - r/c}$ = retarded time

$$\boxed{\begin{aligned}\vec{E}(\vec{r},t) &= \frac{\mu_0}{4\pi r}\left[\left(\hat{r}\cdot\ddot{\vec{p}}(t_0)\right)\hat{r} - \ddot{\vec{p}}(t_0)\right] \\ &= \frac{\mu_0}{4\pi r}\,\ddot{p}(t_0)\sin\theta\,\hat{\theta}\end{aligned}}$$



$\hat{r}\times(\ddot{\vec{p}}\times\hat{r})$ is in $\hat{\theta}$ direction

Similarly $\vec{B}(\vec{r},t) = \int d\omega\,\vec{B}(\vec{r},\omega)\,e^{-i\omega t}$

$\ddot{p} = |\ddot{\vec{p}}|$

$$= \frac{-\mu_0}{4\pi c r}\int d\omega\, e^{-i\omega(t-r/c)}\,\omega^2\,\vec{p}(\omega)\times\hat{r}$$

$$= \frac{\mu_0}{4\pi c r}\,\frac{\partial^2}{\partial t^2}\left[\int d\omega\, e^{-i\omega(t-r/c)}\,\vec{p}(\omega)\times\hat{r}\right]$$

$$\boxed{\begin{aligned}\vec{B}(\vec{r},t) &= \frac{\mu_0}{4\pi c r}\,\ddot{\vec{p}}(t_0)\times\hat{r} \\ &= \frac{\mu_0}{4\pi c r}\,\ddot{p}(t_0)\sin\theta\,\hat{\phi}\end{aligned}}$$