

Potentials from a moving point charge

Liénard - Wiechart potentials

In general

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \int dt' \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \int dt' \frac{\vec{j}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

For a pt charge moving on trajectory $\vec{r}_0(t)$ with velocity $\vec{v}(t) = \frac{d\vec{r}_0}{dt}$

$$\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_0(t))$$

Substitute in

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \int dt' \frac{q \delta(\vec{r}' - \vec{r}_0(t'))}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

do $\int d^3r'$ using $\delta(\vec{r}' - \vec{r}_0(t'))$ then

$$= \frac{1}{4\pi\epsilon_0} \int dt' \frac{q}{|\vec{r} - \vec{r}_0(t')|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c}\right)$$

now we want to do $\int dt'$ using the $\delta\left(t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c}\right)$

the δ -function is of the form $\delta(g(t'))$ where

$$g(t') = t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

to evaluate, change variable of integration from t' to g

$$\int_{-\infty}^{\infty} dt' f(t') \delta(g(t')) = \int_{g(-\infty)}^{g(\infty)} f(t') \delta(g) \left(\frac{dt'}{dg} \right) dg$$

$$= f(t') \frac{dt'}{dg} = \frac{f(t')}{\left(\frac{dg}{dt'} \right)} \quad \text{evaluated at } t' \text{ such that } g(t') = 0$$

Here: $f(t') = \frac{q}{|\vec{r} - \vec{r}_0(t')|}$

$$t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c} \quad \text{is the retarded time}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c} \frac{d}{dt} |\vec{r} - \vec{r}_0(t')|$$

$$\frac{d}{dt} \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = \frac{[(x-x_0) \frac{dx_0}{dt} + (y-y_0) \frac{dy_0}{dt} + (z-z_0) \frac{dz_0}{dt}]}{\sqrt{\quad}}$$

$$= \frac{-(\vec{r} - \vec{r}_0(t')) \cdot \vec{v}(t')}{|\vec{r} - \vec{r}_0(t')|}$$

$$\equiv -\hat{n}(t') \cdot \vec{v}(t')$$

$$\text{where } \hat{n}(t') \equiv \frac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$$

unit vector from charges position, at retarded time t' , to observer at position r , time t .

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{r}_0(t')|} \frac{1}{1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t')}$$

Liénard-Wiechert
potentials

similarly:

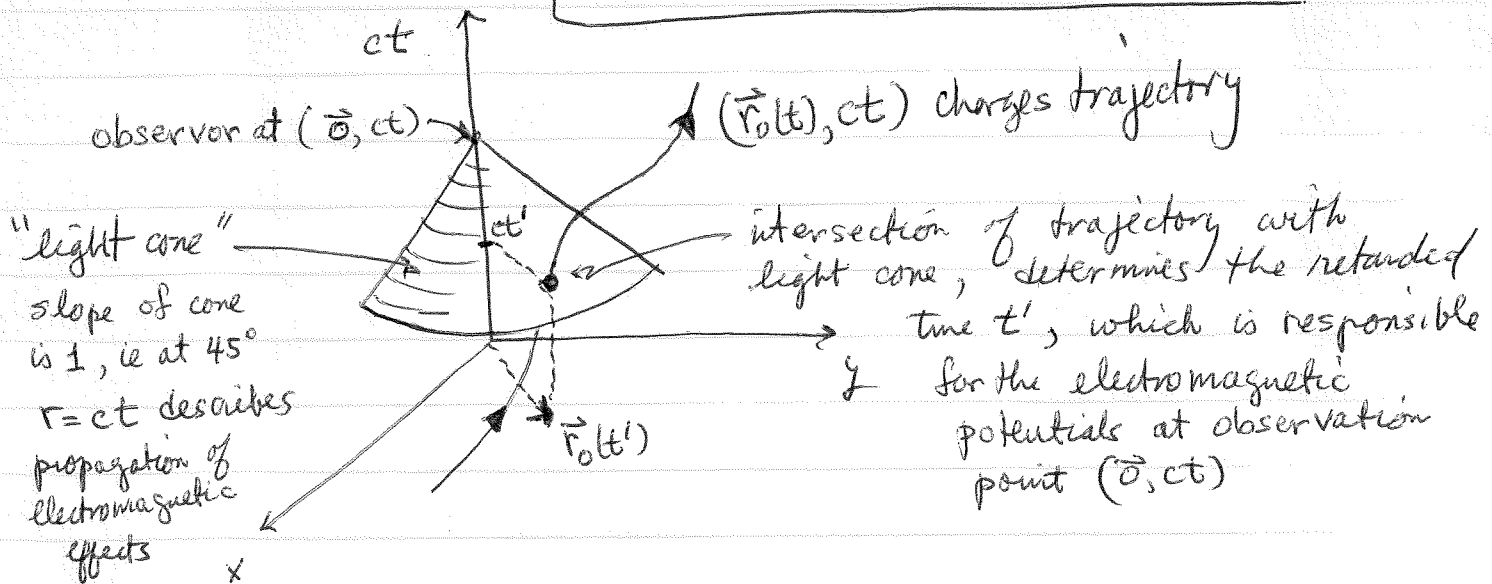
$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q \vec{v}(t')}{|\vec{r} - \vec{r}_0(t')|} \frac{1}{1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t')}$$

$$t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

using $\mu_0 = \frac{1}{\epsilon_0 c^2} \Rightarrow$

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}(t')}{c^2} V(\vec{r}, t)$$

graphically:



what is observed at the apex of the cone, at point $(\vec{0}, ct)$ can only be caused by events which occurred somewhere on the surface of the light cone.

pts r' on light cone determined by eqn $ct' = ct - r'$ as observer at $\vec{r} = \vec{0}$
 $\Rightarrow r' = ct - ct'$

Potential from charge moving with constant velocity

$$\vec{r}_0(t) = \vec{v}t$$

find retarded time $t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c} \Rightarrow c(t-t') = |\vec{r} - \vec{r}_0(t')|$

$$\begin{aligned} \rightarrow c^2(t-t')^2 &= |\vec{r} - \vec{r}_0(t')|^2 = r^2 + v^2 t'^2 - 2\vec{r} \cdot \vec{v} t' \\ c^2 t^2 + c^2 t'^2 - 2c^2 t t' &= r^2 + v^2 t'^2 - 2\vec{r} \cdot \vec{v} t' \end{aligned}$$

quadratic eqn in t'

$$\rightarrow (c^2 - v^2)t'^2 - 2(c^2 t - \vec{r} \cdot \vec{v})t' + c^2 t^2 - r^2 = 0$$

quadratic formula

$$t' = \frac{(c^2 t - \vec{r} \cdot \vec{v}) \pm \sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 - (c^2 t^2 - r^2)(c^2 - v^2)}}{(c^2 - v^2)} \quad (*)$$

to get sign correct, note that when $v=0$, we want

$$t' = t - \frac{r}{c}$$

Apply to above:

$$t' = \frac{c^2 t \pm \sqrt{c^4 t^2 - c^4 t^2 + c^2 r^2}}{c^2}$$

$$= t \pm \frac{r}{c} \Rightarrow \text{take } \ominus \text{ sign}$$

scalar potential $V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0(t')| (1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t'))}$

here $\vec{v}(t') = \vec{v}$ constant

$$\begin{aligned} \underbrace{|\vec{r} - \vec{r}_0(t')|}_{= c(t-t')} \left(1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}\right) &= \underbrace{|\vec{r} - \vec{v}t'|}_{= c(t-t')} \left(1 - \frac{1}{c} (\vec{r} - \vec{v}t') \cdot \vec{v}\right) \end{aligned}$$

$$= c(t-t') \left[1 - \frac{\vec{r} \cdot \vec{v} - v^2 t'}{c(t-t')} \right]$$

$$= c(t-t') - \frac{1}{c} \vec{r} \cdot \vec{v} + \frac{v^2 t'}{c} = ct - \frac{\vec{r} \cdot \vec{v}}{c} - c(1 - \frac{v^2}{c^2})t'$$

$$= \frac{1}{c} [c^2 t - \vec{r} \cdot \vec{v} - (c^2 - v^2) t'] \quad \text{insert (*) for } t'$$

$$= \frac{1}{c} [c^2 t - \vec{r} \cdot \vec{v} - (c^2 t - \vec{r} \cdot \vec{v}) + \sqrt{(c^2 t^2 - \vec{r} \cdot \vec{v})^2 - (c^2 t - r^2)(c^2 - v^2)}]$$

$$= \frac{1}{c} \sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 - (c^2 t^2 - r^2)(c^2 - v^2)}$$

simplify ~~the~~ square root

$$= \frac{1}{c} \sqrt{c^4 t^2 + (\vec{r} \cdot \vec{v})^2 - 2c^2 t \vec{r} \cdot \vec{v} - c^4 t^2 + r^2 c^2 + c^2 v^2 t^2 - r^2 v^2}$$

$$= \frac{1}{c} \sqrt{c^2 (\vec{r} - \vec{v}t)^2 + (\vec{r} \cdot \vec{v})^2 - r^2 v^2}$$

$$= \sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r} \cdot \vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}$$

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2 + \frac{(\vec{r} \cdot \vec{v})^2}{c^2} - \frac{r^2 v^2}{c^2}}}$$

Note, for $(\frac{v}{c})^2 \ll 1$

$$V(\vec{r}, t) \approx \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(\vec{r} - \vec{v}t)^2}} = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}_0(t)|^2}$$

"quasi static" limit
looks like instantaneous
Coulomb potential

vector potential

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2} V(\vec{r}, t)$$

fields from charges moving with constant \vec{v}

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V - \frac{\vec{v}}{c^2} \frac{\partial V}{\partial t} \quad \text{as } \vec{A} = \frac{\vec{v}}{c^2} V$$

$$-\vec{\nabla}V = \frac{q}{4\pi\epsilon_0} \frac{1}{r^3} \left[\vec{\nabla}(\vec{r} - \vec{v}t)^2 + \frac{1}{c^2} \vec{v}(\vec{r} \cdot \vec{v})^2 - \frac{v^2}{c^2} \vec{\nabla}r^2 \right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r^3} \left[\vec{r} - \vec{v}t + \frac{\vec{v}}{c^2} (\vec{r} \cdot \vec{v}) - \frac{v^2}{c^2} \vec{r} \right]$$

$$-\frac{\partial V}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^3} \left[\frac{\partial}{\partial t} (\vec{r} - \vec{v}t)^2 \right] = \frac{q}{4\pi\epsilon_0} \frac{(-1)}{r^3} \left[(\vec{r} - \vec{v}t) \cdot \vec{v} \right]$$

$$\vec{E} = -\vec{\nabla} - \frac{\vec{v}}{c^2} \frac{\partial V}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^3} \left[\vec{r} - \vec{v}t + \frac{\vec{v}}{c^2} (\vec{r} \cdot \vec{v}) - \frac{v^2}{c^2} \vec{r} - \frac{\vec{v}}{c^2} (\vec{r} \cdot \vec{v} - v^2 t) \right]$$

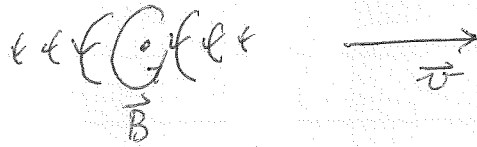
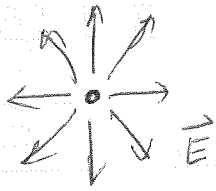
$$\boxed{\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{v}t)(1 - v^2/c^2)}{\left((\vec{r} - \vec{v}t)^2 + (\frac{\vec{r} \cdot \vec{v}}{c})^2 - (\frac{rv}{c})^2 \right)^{3/2}}$$

$$\vec{B} = \vec{v} \times \vec{A} = \nabla \times \left(\frac{\vec{v}}{c^2} V \right) = (\vec{\nabla}V) \times \frac{\vec{v}}{c^2} = \left(-\vec{E} - \frac{\partial \vec{A}}{\partial t} \right) \times \frac{\vec{v}}{c^2}$$

$$= -\vec{E} \times \frac{\vec{v}}{c^2} - \frac{\partial}{\partial t} \left(\frac{\vec{v}}{c^2} V \right) \times \frac{\vec{v}}{c^2} = \frac{\vec{v}}{c^2} \times \vec{E} - \frac{\partial V}{\partial t} \underbrace{\frac{\vec{v}}{c^2} \times \frac{\vec{v}}{c^2}}_{=0}$$

$$\boxed{\vec{B}(\vec{r}, t) = \frac{\vec{v}}{c^2} \times \vec{E}(\vec{r}, t)}$$

Note: The factor $1 - v^2/c^2$ reminds one of special relativity!



for $v^2 \ll c^2$

$$\vec{E}(\vec{r}, t) \approx \frac{q}{4\pi\epsilon_0} \frac{(\vec{r} - \vec{v}t)}{|\vec{r} - \vec{v}t|^3} \quad \text{instantaneous Coulomb field}$$

$$\begin{aligned} \vec{B}(\vec{r}, t) &= \frac{\vec{v}}{c^2} \times \vec{E} = \frac{1}{4\pi\epsilon_0 c^2} q \frac{\vec{v} \times (\vec{r} - \vec{v}t)}{|\vec{r} - \vec{v}t|^3} \\ &= \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \end{aligned}$$

$$\text{with } \vec{j}(\vec{r}') = q \vec{v} \delta(\vec{r}' - \vec{v}t)$$

so \vec{B} looks just like Biot-Savart law, even though for moving charge, $\vec{v} \cdot \vec{j} \neq 0$.

Since a charge moving with constant \vec{v} can always be considered to be at rest in another ~~frame~~ frame of reference, the formulas above for \vec{E} and \vec{B} for charge moving with \vec{v} , will tell us how \vec{E} and \vec{B} transform under a Lorentz transformation between two frames in relative uniform motion. ~~Thus~~ In fact, the Lorentz transf so defined, was first discovered as a law of electrodynamics, before the theory of special relativity was discovered.

For a pt charge in state of arbitrary motion ($\dot{\vec{v}} \neq 0$)
 one finds (see text)

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r} - \vec{r}_0(t')}{\left([\vec{r} - \vec{r}_0(t')] \cdot [c\hat{m}(t') - \vec{v}(t')] \right)^3} \left[[c\hat{m}(t') - \vec{v}(t')] (c^2 - v^2) + (\vec{r} - \vec{r}_0(t')) \times ([c\hat{m}(t') - \vec{v}(t')] \times \vec{a}(t')) \right]$$

where $t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c}$ is retarded time

$$\hat{m}(t') = \frac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|} \quad \vec{v} = \frac{d\vec{r}_0}{dt} \quad , \quad \vec{a} = \frac{d^2\vec{r}_0}{dt^2}$$

first term $\sim \frac{1}{r^2}$ is "generalized Coulomb field" or "velocity field"

second term $\sim \frac{1}{r}$ is "acceleration field"

$$\vec{B}(\vec{r}, t) = \frac{\hat{m}(t')}{c} \times \vec{E}(\vec{r}, t)$$

\vec{B} always $\perp \vec{E}$, and $\vec{B} \perp \hat{m}$, vector from observer at \vec{r}
~~at~~ to retarded point $\vec{r}_0(t')$.

Can use above to write down total force between
 two point charges, in arbitrary states of motion (see text)

Can use to construct \vec{S} and get power radiated by an
 accelerating point charge - in non-relativistic limit,
 one recovers Larmor's formula - see text