Inertial frames of reference: Set of frames of reference which move at constant velocity with respect to each other.

Special Relativity

1) Speed of light is constant in all inertial frames of reference.
2) Physical laws must look the same in all inertial frames of reference - there is no experiment that can determine the "absolute" velocity of any inertial frame.

⇒ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront is \( r^2 - c^2 t^2 = 0 \)

⇒ \((x, y, z, t)\) coords in one inertial frame \(K\)
\((x', y', z', t')\) coords in another inertial frame \(K'\) that moves with velocity \(v = v/\gamma\) with respect to \(K\).

What is the transformation that relates coords in \(K'\) to coords in \(K\)?

\[ y = y', \quad z = z' \] (origins of \(K\) and \(K'\) coincide when \(t = t' = 0\))

\[ c^2 t^2 - x^2 = c^2 t'^2 - x'^2 \]

⇒ \( \frac{(ct + x)}{(ct' + x')} \frac{(ct - x)}{(ct' - x')} = 1 \)

Expect transformation to be linear.

⇒ \( ct' + x' = (ct + x) f \)
\( ct' - x' = (ct - x) f^{-1} \)

for some constant \(f\).

Write \( f = e^{-\gamma} \)
Solve for $ct'$ and $x'$ in terms of $ct$ and $x$

\[
ct' = ct \left( \frac{e^y + e^{-y}}{2} \right) - x \left( \frac{e^y - e^{-y}}{2} \right)
\]

\[
x' = -ct \left( \frac{e^y - e^{-y}}{2} \right) + x \left( \frac{e^y + e^{-y}}{2} \right)
\]

\[
ct' = ct \cosh y - x \sinh y
\]

\[
x' = -ct \sinh y + x \cosh y
\]

**Meaning of parameter $y$**

*(at $x = 0$)*

The origin of $K$ has trajectory $x' = -v t'$ in $K'$

\[
\implies \frac{x'}{ct'} = -v
\]

From transformation above, with $x = 0$, we get

\[
\frac{x'}{ct'} = -ct \frac{\sinh y}{\cosh y} = -\tanh y
\]

So \( \frac{v}{c} = \tanh y \)

\[
\implies \cosh y = \frac{1}{\sqrt{1 - \left( \frac{v}{c} \right)^2}} = \beta
\]

\[
\sinh y = \frac{v}{c} \beta
\]

**Lorentz Transformation**

\[
\begin{align*}
ct' &= x \beta - \beta \frac{v}{c} x \\
x' &= -\beta \frac{v}{c} ct + \beta x
\end{align*}
\]
An inverse Lorentz transform obtained by Tabor:

\[ \nu \rightarrow -\nu \] as above.

\[ \begin{align*}
    ct &= \gamma ct' + \gamma \left( \frac{u}{c} \right) x' \\
x &= \gamma \left( \frac{u}{c} \right) ct' + \delta x'
\end{align*} \]
Time Dilation

Consider a clock located at the origin in frame $K'$ that moves with velocity $v \hat{x}$ as seen from "lab" frame $K$.

The clock in frame $K'$ ticks at $t'_1 = 0$ and $t'_2 = T_0$.

Time between ticks in frame $K'$ is thus $T_0$.

What is time between ticks in frame $K$?

Since clock 1 is at origin of $K$, then position of 1st and 2nd ticks in $K$ is at $x'_1 = x'_2 = 0$.

In frame $K$, the observer sees tick 1 at

$$ct_1 = x'_1 + y'(\frac{v}{c})x'_1 = 0 + 0 = 0$$

and tick 2 at

$$ct_2 = x'_2 + y'(\frac{v}{c})x'_2 = ycT_0 + 0 = ycT_0$$

So $t_1 = 0$, $t_2 = \gamma T_0$

So time between ticks as seen by $K$ is $\Delta t = \gamma T_0 > T_0$.

So it looks to $K$ as if $K'$'s clock has slowed down.

**Proper Time** - time between two events as measured in the frame of reference in which those two events occur at the same position.
Fitzgerald Contraction

Consider frame $K'$ moving with $v < c$ as seen by $K$. A ruler, at rest in $K'$, has its ends located at $x_1' = 0$, $x_2' = L_0'$. What is the length $L'$ of the ruler as seen by $K$?

At $t = 0$ in frame $K$, the observer measures the positions of the two ends of the ruler and finds

$$x_1' = 0 = -\gamma \left(\frac{v}{c}\right) c t_1 + \gamma x_1 = 0 + \gamma x_1,$$

$$\Rightarrow x_1 = 0$$

and

$$x_2' = L_0' = -\gamma \left(\frac{v}{c}\right) c t_2 + \gamma x_2 = 0 + \gamma x_2.$$

$$\Rightarrow x_2 = \frac{L_0'}{\gamma}$$

So length of ruler in $K$ is $x_2 - x_1 = \frac{L_0'}{\gamma} < L_0$.

It appears to $K$ as if the ruler has contracted.

Proper length - distance between two events as measured in the frame in which the two events happen at the same time.
Note: K's measurement of left end occurs at time:
\[ ct_1' = \gamma ct_1 - \gamma \left( \frac{v}{c} \right) x_1 = 0 \quad \Rightarrow \quad t_1' = 0 \]

K's measurement of right end occurs at time:
\[ ct_2' = \gamma ct_2 - \gamma \left( \frac{v}{c} \right) x_2 = 0 - \gamma \left( \frac{v}{c} \right) \frac{L_0}{\gamma} = \frac{-v}{c} L_0 \]
\[ t_2' = -\frac{v}{c} L_0 \]

So K's interpretation of K's measurement is that K first measures the position of the right end of the ruler, and only a true \( \frac{v}{c} \) later measures the location of the left end.

So K' sees K measure a length
\[ L_0 - \frac{v^2}{c^2} L_0 \]
\[ \text{duration ruler travels between K's two measurements} \]
\[ = L_0 \left( 1 - \frac{v^2}{c^2} \right) = \frac{L_0}{\gamma^2} \]

So K's two measurements, which are simultaneous to K, do not occur simultaneously to K'.

Events that are simultaneous in one frame of reference are not simultaneous in another frame of reference.
So $K'$ sees $K$ measure a length that is according to $K'$ a length equal to $\frac{L_0}{\gamma^2}$.

But $K'$ also sees that $K$ is measuring with a contracted length scale that is $\sqrt{1 - \gamma^2}$ smaller. So the length $L_0$ seen by $K'$ looks like the length

$$\left(\frac{L_0}{\gamma^2}\right) \frac{1}{(\gamma^2)}$$

when $K'$ sees $K$ measure it with $K'$'s contracted ruler. This $K'$ will agree that $K$ thinks the ruler is $\frac{L_0}{\gamma} = \frac{L_0}{\gamma}$ long.

$K$ thinks the moving ruler has contracted.
$K'$ thinks $K$ is both (i) not measuring the ruler at the same time $t_0$ as (ii) measuring the length of $K'$'s ruler with $K$'s contracted ruler.

So they cannot both agree on the outcome of what happens, but they describe different physical processes to what is happening.
two events \((x_1, t_1)\) seen in \(K\) and 
\((x_2, t_2)\)

Transform to frame \(K'\) in which they are at
same position \(x_1' = x_2'\). The two \(t_2' - t_1'\) in \(K'\) is time
between the events.

\[
\begin{align*}
ct_1' &= \gamma ct_1 - \gamma \left(\frac{v}{c}\right)x_1 \\
ct_2' &= \gamma ct_2 - \gamma \left(\frac{v}{c}\right)x_2
\end{align*}
\]

\[
\begin{align*}
x_1' &= -\gamma \left(\frac{v}{c}\right)ct_1 + \gamma x_1 \\
x_2' &= -\gamma \left(\frac{v}{c}\right)ct_2 + \gamma x_2
\end{align*}
\]

\[
x_1' = x_2' \Rightarrow \gamma (x_2 - x_1) - \gamma \left(\frac{v}{c}\right)c(t_2 - t_1) = 0
\]

\[
\Rightarrow \frac{x_2 - x_1}{t_2 - t_1} = \frac{v}{c}
\]

So frame \(K'\) travels with \(\frac{v}{c}\) with respect to \(K\).

Clearly can have such at \(K'\) only if \(v < c\).

The time difference between the events in \(K'\) is

\[
t_2' - t_1' = \gamma t_2 - \gamma \left(\frac{v}{c}\right)x_2 - \gamma t_1 + \gamma \left(\frac{v}{c}\right)x_1
\]

\[
= \gamma \left( t_2 - t_1 + \frac{v}{c^2} (x_2 - x_1) \right)
\]

\[
= \gamma (t_2 - t_1 - \frac{v^2}{c^2} (t_2 - t_1))
\]

\[
= (t_2 - t_1) \gamma (1 - \frac{v^2}{c^2}) = (t_2 - t_1) \gamma / c^2
\]

\[
[2' = \frac{t_2' - t_1'}{\gamma}]
\]
Proper length

two events \((x_1, t_1) (x_2, t_2)\) seem in \(K\) transform to \(K'\) in which the occur at same time \(t_1' = t_2'\). The distance \(x_2' - x_1'\) in that frame \(K'\) is the proper length between the two events.

\[ x_1' = -\gamma \left(\frac{v}{c}\right) c t_1 + \gamma x_1, \]
\[ x_2' = -\gamma \left(\frac{v}{c}\right) c t_2 + \gamma x_2. \]

\[ c t_1' = \gamma c t_1 - \gamma \left(\frac{v}{c}\right) x_1, \]
\[ c t_2' = \gamma c t_2 - \gamma \left(\frac{v}{c}\right) x_2. \]

\(t_1' = t_2' \Rightarrow \gamma c (t_2 - t_1) - \gamma \left(\frac{v}{c}\right) (x_2 - x_1) = 0\)

\[ \frac{x_2 - x_1}{t_2 - t_1} = \frac{c^2}{v^2} \]

or \(v = \frac{c^2 (t_2 - t_1)}{(x_2 - x_1)}\) such a frame \(K'\) can exist only if \(v < c\) or \(\frac{x_2 - x_1}{t_2 - t_1} > c\)

Then the proper length is

\[ l \equiv x_2' - x_1' = \gamma (x_2 - x_1) - \gamma \left(\frac{v}{c}\right) c (t_2 - t_1) \]
\[ = \gamma (x_2 - x_1) - \gamma \left(\frac{v}{c}\right) c \frac{v}{c^2} (x_2 - x_1) \]
\[ = (x_2 - x_1) \gamma (1 - \frac{v^2}{c^2}) = (x_2 - x_1) \gamma / \gamma' \]

\[ l = \frac{x_2 - x_1}{\gamma} \]
Consider two events, one of which occurs at \((x_1 = 0, t_1 = 0)\) and the other at \((x_2, t_2)\).

The time-like region \( \frac{x_2}{t_2} < c \) consists of all points such that there is a frame in which \( x_2 \) occurs at the same position as \( x_1 \) and we can therefore define the proper time between the two events.

Time-like region is such that a pulse of light emitted at origin at \( t_1 = 0 \) will arrive at position \( x_2 \) at a time earlier than \( t_2 \).

The space-like region \( \frac{x_2}{t_2} > c \) consists of all points such that there is a frame in which \( t_2 \) occurs at the same time as \( t_1 \), and we can therefore define the proper length between the two events.

Space-like region is such that a pulse of light emitted at origin at \( t_1 = 0 \) will arrive at position \( x_2 \) at a time later than \( t_2 \).

The light cone \( \frac{x_2}{t_2} = c \) separates the time-like from the space-like regions. The light cone at origin can affect only events in its future time-like region. It is affected only by events in its past time-like region.
Inverse transform obtained by taking $\nu \to -\nu$ as above

\[
\begin{align*}
ct &= \gamma ct' + \gamma (\frac{\nu}{c}) x' \\
x &= \gamma (\frac{\nu}{c}) ct' + \gamma x'
\end{align*}
\]

$\gamma$ - vectors

$c$ position: $x_\mu = (x_1, x_2, x_3, ct)$ $x_4 = ct$

sum over repeated indices Lorentz invariant scalar

Lorentz transformation $x_\mu' = \sum_{\nu=1}^{4} \gamma_{\mu\nu} x_\nu$ $x_4' = ct'$

$L$ matrix of Lorentz transformation $L$

\[
\alpha(L) = \begin{pmatrix}
\gamma & 0 & 0 & \frac{i \nu}{c} x  \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\frac{i \nu}{c} x & 0 & 0 & \gamma
\end{pmatrix}
\]

Inverse: $x_\mu = a_{\mu\nu}(L^{-1}) x_\nu$

$a_{\mu\nu}(L^{-1})$ is given by taking $\nu \to -\nu$ in $a_{\mu\nu}(L)$

we see $a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$ inverse = transpose $= "\text{orthogonal}"$