More generally

Since $x^2_\mu$ is Lorentz invariant scalar,

$$x^2_\mu = a_{\mu\nu}(L) a^{\mu\nu}(L) x_\nu x_\lambda = x^2$$

$$\Rightarrow a_{\mu\nu}(L) a^{\mu\nu}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a_{\mu\nu}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a^{\mu\nu} = a_{\mu\nu}(L)^{-1} = \text{transpose} = \text{inverse}$$

A matrix whose transpose equals its inverse is called an orthogonal matrix.

If $L_1$ is a Lorentz transform from $K$ to $K'$

$L_2$ is a Lorentz transform from $K'$ to $K''$.

Then the Lorentz transform from $K$ to $K''$ is given by the matrix

$$a(L_2 L_1) = a(L_2) a(L_1)$$

If $L_1 = L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow a^{-1}(L) = a(L^{-1})$$

Particle on trajectory $x_1 = v_1(t + dt) - x_1(t)$

$$dx_\mu = (dx_1, dx_2, dx_3, i c dt)$$

$$-(dx_\mu)^2 = c^2 ds^2 = c^2 dt^2 - dr^2$$ Lorentz invariant scalar

$$ds^2 = \frac{dt^2}{y^2} \left[ 1 - \frac{1}{c^2} \left( \frac{dx_1}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_2}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_3}{dt} \right)^2 \right]$$

$$ds^2 = \frac{dt^2}{y^2} \quad \vec{\nu} = \frac{dx}{dt}$$

proper time interval
A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does \(\chi_\mu\).

**4-velocity**

\[
U^\mu = \frac{d\chi^\mu}{ds} = \chi_\mu
\]

since \(d\chi^\mu\) is a 4-vector and \(ds\) is Lorentz invariant scalar, then \(\frac{d\chi_\mu}{ds}\) is a 4-vector.

Space components \(\vec{U} = \gamma \vec{V}\)

\[
U_\mu = i c \gamma
\]

\[
U_\mu U_\mu = \gamma^2 v^2 - c^2 y^2 = \gamma^2 (v^2 - c^2) = \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2 \text{ Lorentz invariant scalar}
\]

**4-acceleration**

\[
\alpha^\mu = \frac{dU^\mu}{ds} = \gamma \frac{dU^\mu}{dt}
\]

**4-gradient**

\[
\nabla = \left( \vec{\nabla}, -\frac{1}{c^2} \frac{\partial}{\partial t} \right) = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right)
\]

proof \(\frac{\partial}{\partial x_\mu}\) is a 4-vector

by chain rule:

\[
\frac{\partial}{\partial x_\mu} = \frac{\partial X_\lambda}{\partial x_\mu} \frac{\partial}{\partial X_\lambda} \rightarrow \text{ but } \frac{\partial X_\lambda}{\partial x_\mu} = A_{\mu \lambda} (L^{-1})
\]

So

\[
\frac{\partial}{\partial x_\mu} = A_{\mu \lambda} (L) \frac{\partial}{\partial X_\lambda}
\]

So transforms same as \(X_\mu\)

\[
\left( \frac{\partial}{\partial x_\mu} \right)^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \quad \text{wave equation operator}
\]

**Inner products**

If \(U^\mu\) and \(V^\mu\) are 4-vectors, then \(U^\mu V^\mu\) is Lorentz invariant scalar
Maxwell's Equation's in Relativistic Form
How do \(E\) and \(B\) transform under Lorentz Transformation?
\(E\) and \(B\) have much more complicated transformation laws than position 4-vector \(\mathbf{x}\) or 4-momentum \(p_{\mu} = (E, \mathbf{p})\)

Example: parallel plate capacitor at rest in \(K\)
plates have area \(A\), charge \(Q\)

\[
\mathbf{E} = \frac{Q}{AE_0} \hat{\mathbf{y}} \quad \text{uniform} \quad \frac{Q}{A} = \sigma \text{ surface charge den.}
\]
\[
\mathbf{B} = 0
\]

In \(K'\), moving with \(\mathbf{v} = v \hat{\mathbf{y}} \) wrt \(K\), \(y\) dimension
of plates is contracted by factor \(\gamma = \frac{1}{\sqrt{1 - v^2}}\) (Fitzgerald Contraction)

\[
\sigma' = \frac{Q}{A'} = \frac{Q}{AY} = \frac{Q}{A} = \sigma
\]
\[
\mathbf{E}' = \frac{Q}{A'E_0} \hat{\mathbf{y}} = \frac{Q}{AE_0} \hat{\mathbf{y}} = \gamma \mathbf{E}
\]
\(\mathbf{E}\) is along \(\hat{\mathbf{y}} \perp \mathbf{v}\).

This is different than transverse law for \(\mathbf{E}\).
Under L.T., components of \(\mathbf{E} \perp \mathbf{v}\) do not change
But components of \(\mathbf{E} \parallel \mathbf{v}\) do change.

Also, moving surface charge \(\sigma'\) gives rise to surface current density \(\mathbf{j}\) \(\Rightarrow\) there will be magnetic field \(\mathbf{B}'\) in frame \(K'\) \(\Rightarrow\) Lorentz trans. must couple together the components of \(\mathbf{E}\) and \(\mathbf{B}\).
Electromagnetism

Clearly $E + B$ must transform into each other under Lorentz transform.

In inertial frame $K$
In frame $K'$ moving with $v \parallel E$ to cage stationary line charge $\lambda$

\[ E = \frac{\lambda}{c} \]

\[ \nabla \times B \]

Cylindrical outward electric field
and $B$-field

Moving line charge gives current $= B$ circulating around wire as well as outward radial $E$

Lorentz force
\[ \vec{F} = q \vec{E} + q \vec{v} \times \vec{B} \]

What is the velocity $\vec{v}$ here? Velocity with respect to what inertial frame? Clearly $E$ and $B$ must change from inertial frame to another if this force law can make sense.

Charge density, current density

\[ \Delta V \text{ contains charge } \Delta Q \]

Consider charge $\Delta Q$ contained in a vol $\Delta V$.
$\Delta Q$ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame
\[ \Delta Q = \vec{\rho} \Delta V \quad \vec{\rho} \text{ is charge density in rest frame of charge} \]
\[ \Delta V \text{ is volume of box in rest frame} \]
\[ \vec{\rho} \text{ is a Lorentz invariant scalar by definition} \]

Now transform to another frame moving with velocity \( \vec{v} \) with respect to the rest frame.

\[ \Delta Q \text{ remains the same,} \]
\[ \Delta V = \frac{\Delta Q}{\Delta \gamma} \quad \text{volume contracts in direction \( \gamma \) to} \vec{v} \]
\[ \Rightarrow \vec{\rho} = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \gamma} \quad \gamma = \sqrt{1 - \frac{v^2}{c^2}} \text{ spatial components} \]

\[ \vec{\gamma} = \vec{\rho} \vec{\gamma} = \left( \frac{\vec{\rho}}{\gamma} \right) \left( \gamma \vec{\gamma} \right) = \vec{\rho} \vec{\gamma} \]

current density \( \vec{\gamma} \)

Define 4-current \( \vec{j} = \vec{\rho} \lambda \mu = \vec{\rho} \left( \vec{\gamma}, ic \gamma \right) \)

spatial components of \( \vec{j} \) are \( \vec{j} = \vec{\rho} \vec{\gamma} = \vec{\rho} \vec{\gamma} \) current density

temporal component of \( \vec{j} \) is \( \vec{j} \mu = ic \vec{\rho} \gamma \gamma = ic \vec{\rho} \) charge density

So \[ \vec{j} \mu = (\vec{j}, ic \vec{\rho}) \]

\( \vec{j} \mu \) is a 4-vector since \( \mu \) is a 4-vector and \( \vec{\rho} \) is Lorentz invariant scalar

length of the 4-current is \[ \vec{j} \cdot \vec{j} \mu = \vec{j}^2 - c^2 \vec{\rho}^2 = \vec{\rho}^2 \lambda \mu \mu \]

\[ = -c^2 \vec{\rho}^2 \]

Charge conservation \[ 0 = \vec{\nabla} \cdot \vec{j} + \frac{\partial \vec{\rho}}{\partial t} = \vec{\nabla} \cdot \vec{j} + \frac{\partial (ic \vec{\rho})}{\partial (ict)} = \vec{\nabla} \cdot \vec{j} + \frac{\partial \vec{j} \mu}{\partial t} \]

\[ \Rightarrow \frac{\partial \vec{j} \mu}{\partial t} = 0 \]

charge conservation in Lorentz covariant form
Equations for potentials in Lorentz gauge

\[(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{A} = \square \vec{A} = -\mu_0 \vec{J} \]
\[c^2 = \frac{1}{\mu_0 \varepsilon_0} \]

\[(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \nabla V = \square \nabla V = -\frac{\rho}{\varepsilon_0} = -c^2 \mu_0 \varepsilon_0 \]

So

\[\square \vec{A} = -\mu_0 \vec{J} \] \[= -\mu_0 (ic \phi) \left( \frac{\varepsilon}{\mu} \right) \]

\[\square (i \nabla \phi) = -\mu_0 \vec{J} \cdot \vec{E} \]

Define 4-potential \[A_\mu = (\vec{A}, i \nabla \phi)\]

\[\Rightarrow \square A_\mu = -\mu_0 \vec{J} \cdot \vec{E} \] equation for potentials

\[\square \phi \phi \frac{\partial^2}{\partial x^2} \text{ in Lorentz invariant operator} \]

So we can write the above as

\[\frac{\partial^2 A_\mu}{\partial x^2} = -\mu_0 \vec{J} \cdot \vec{E} \]

Lorentz gauge condition is

\[0 = \nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \nabla \cdot \vec{E} = \nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \nabla}{\partial t} \]

\[= \nabla \cdot \vec{A} + \frac{\partial (i \nabla \phi)}{\partial (ict)} = \nabla \cdot \vec{A} + \frac{\partial A_\mu}{\partial x} \]

\[= \frac{\partial A_\mu}{\partial x} \mu \]

So Lorentz gauge condition is

\[\frac{\partial A_\mu}{\partial x} = 0 \]
Electric and Magnetic Fields

\[ \mathbf{B} = \nabla \times \mathbf{A} \quad \Rightarrow \quad B_i = \frac{\partial A_k - \partial A_j}{\partial x_j - \partial x_k} \]

where \( i, j, k \) are a cyclic permutation of 1, 2, 3

\[ \mathbf{E} = \nabla \mathbf{A} - \frac{\partial \mathbf{A}}{\partial t} \]

\[ V = \frac{c A_y}{i}, \quad x_4 = i c t \]

\[ \Rightarrow \quad E_i = -\partial \left( \frac{c A_y}{i} \right) - \partial A_i = -\frac{c}{i} \frac{\partial A_y}{\partial x_i} - \frac{c}{i} \frac{\partial A_i}{\partial x_4} \]

\[ \frac{E_i}{i} = \frac{\partial A_y}{\partial x_i} - \frac{\partial A_4}{\partial x_i} \]

has a similar form to \( B_i \)

We define the field strength tensor

\[ F^\mu_\nu = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F^\mu_\nu \quad 4 \times 4 \text{ antisymmetric} \]

\[ F^\mu_\nu = \begin{pmatrix}
  0 & -B_3 & -B_2 & -iE_1/c \\
  B_3 & 0 & -B_1 & -iE_2/c \\
  B_2 & B_1 & 0 & -iE_3/c \\
  iE_1/c & iE_2/c & iE_3/c & 0
\end{pmatrix} \]

"curl" of a 4-vector is a 4x4 antisymmetric 2nd rank tensor

4x4 antisymmetric 2nd rank tensor has only 6 independent components - just the right number to specify the \( \mathbf{E} \) and \( \mathbf{B} \) fields!