More generally

Since $x_\mu'^2$ is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L)$$ transpose = inverse

$a_{\mu\nu}$ is 4×4 orthogonal matrix

~~a matrix whose transpose equals its inverse~~ is called an <u>ortogonal</u> matrix.

If L_1 is a Lorentz transf from K to K'

L_2 is a Lorentz transf from K' to K''

Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow a^{-1}(L) = a(L^{-1})$$

<u>particle on trajectory</u>

$dx_1 = x_1(t+dt) - x_1(t)$

etc

4 - differential

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2$$ Lorentz invariant scalar

$$ds^2 = dt^2\left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2} \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$ proper time interval

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

<u>4-velocity</u> $u_\mu \equiv \frac{dx_\mu}{ds} \equiv \dot{x}_\mu$

$= \gamma \frac{dx_\mu}{dt}$

since dx_μ is a 4-vector and ds is Lorentz invariant scalar, then $\frac{dx_\mu}{ds}$ is a 4-vector,

space components $\vec{u} = \gamma \vec{v}$

$u_4 = ic\gamma$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2(v^2 - c^2)$$

$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$ Lorentz invariant scalar

<u>4-acceleration</u> $a_\mu \equiv \frac{du_\mu}{ds} = \gamma \frac{du_\mu}{dt}$

<u>4-gradient</u> $\frac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \frac{-i}{c}\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right)$

where $x_4 = ict$

proof $\frac{\partial}{\partial x_\mu}$ is a 4-vector

by chain rule: $\frac{\partial}{\partial x_\mu'} = \frac{\partial x_\lambda}{\partial x_\mu'}\frac{\partial}{\partial x_\lambda}$ $\longrightarrow$ but $\frac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1})$

$= a_{\mu\lambda}(L)$

inverse = transpose

So $\frac{\partial}{\partial x_\mu'} = a_{\mu\lambda}(L)\frac{\partial}{\partial x_\lambda}$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$ wave equation operator!

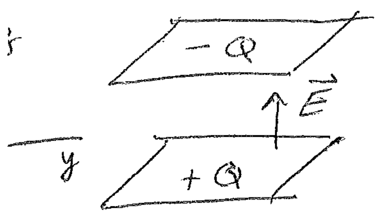
<u>inner products</u>

If U_μ and V_μ are 4-vectors, then $U_\mu V_\mu$ is Lorentz invariant scalar

Maxwell's Equations in Relativistic Form

How do $\vec{E}$ and $\vec{B}$ transform under Lorentz transformation?
$\vec{E}$ and $\vec{B}$ have much more complicated transformation laws than position 4-vector $x_\mu = (\vec{r}, ict)$

Example: parallel plate capacitor at rest in K
plates have area A, charge Q



$$\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z} \quad \text{uniform} \qquad \frac{Q}{A} = \sigma \text{ surface charge den}$$

$$\vec{B} = 0$$

In K', moving with $\vec{v} = v\hat{y}$ wrt K, y dimension of plates is contracted by factor $1/\gamma$ (FitzGerald Contraction)

$$\sigma' = \frac{Q}{A'} = \frac{\gamma Q}{A} = \gamma\sigma$$

$$\vec{E}' = \frac{Q}{A'\varepsilon_0}\hat{z} = \frac{\gamma Q}{A\varepsilon_0}\hat{z} = \gamma\vec{E} \qquad \vec{E} \text{ is along } \hat{z} \perp \vec{v}.$$

This is different than transf law for $\vec{r}$.
Under L.T. components of $\vec{r} \perp \vec{v}$ do not change
But components of $\vec{E} \perp \vec{v}$ do change

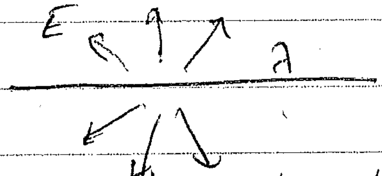
Also, moving surface charge σ' gives rise to surface current density $\Rightarrow$ there will be magnetic field $\vec{B}'$ in frame K'. $\rightarrow$ Lorentz transf must couple together the components of $\vec{E}$ and $\vec{B}$.

<u>Electromagnetism</u>

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

<u>in inertial frame K</u>

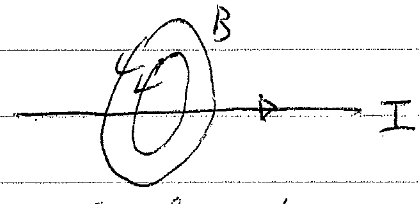
stationary line charge λ



cylindrical outward electric field

no B-field

<u>in frame K' moving with $\vec{v}$ ∥ to wire</u>



moving line charge gives current

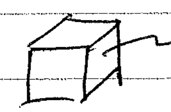
$\Rightarrow$ B circulating around wire as well as outward radial E

<u>Lorentz force</u>

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

<u>Charge density</u>, <u>current density</u>



ΔV contains charge ΔQ

Consider charge ΔQ contained in a vol ΔV.

ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho}\, \Delta \overset{\circ}{V}$ $\overset{\circ}{\rho}$ is charge density in rest frame of charge
$\Delta \overset{\circ}{V}$ is volume of box in rest frame

$\overset{\circ}{\rho}$ is a Lorentz invarient scalar by definition
Now transform to another frame moving with velocity $\vec{v}$
with respect to the rest frame.

ΔQ remains the same,

$\Delta V = \dfrac{\Delta \overset{\circ}{V}}{\gamma}$ volume contracts in direction $\parallel$ to $\vec{v}$

$$\Rightarrow \rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}}\gamma = \overset{\circ}{\rho}\gamma$$

spatial components of 4-velocity

current density is $\vec{j} = \rho\vec{v} = (\rho/\gamma)(\gamma\vec{v}) = \overset{\circ}{\rho}\,\vec{u}$

Define 4-current $j_\mu \equiv \overset{\circ}{\rho}\, u_\mu = \overset{\circ}{\rho}(\vec{u}, ic\gamma)$

spatial components of j_μ are $\vec{j} = \overset{\circ}{\rho}\vec{u} = \rho\vec{v}$ current density

temporal component of j_μ is $j_4 = ic\overset{\circ}{\rho}\gamma = ic\rho$ charge density

So $\boxed{j_\mu = (\vec{j}, ic\rho)}$

j_μ is a 4-vector since
u_μ is a 4-vector and
$\overset{\circ}{\rho}$ is Lorentz invarient scalar

length of the 4-current is $j_\mu j_\mu = |\vec{j}|^2 - c^2\rho^2 = \overset{\circ}{\rho}^2 u_\mu u_\mu$
$= -c^2\overset{\circ}{\rho}^2$

Charge conservation

$$0 = \vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = \vec{\nabla}\cdot\vec{j} + \frac{\partial(ic\rho)}{\partial(ict)} = \vec{\nabla}\cdot\vec{j} + \frac{\partial j_4}{\partial x_4}$$

$$\Rightarrow \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0}$$ charge conservation in Lorentz covariant form

Equations for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \Box^2 \vec{A} = -\mu_0 \vec{j} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = \Box^2 V = -\rho/\varepsilon_0 = -c^2\mu_0 \rho$$

$$= -\mu_0 (ic\rho)\left(\frac{c}{i}\right)$$

$$= -\mu_0 j_4 \left(\frac{c}{i}\right)$$

So

$$\Box^2 \vec{A} = -\mu_0 \vec{j}$$

$$\Box^2 (iV/c) = -\mu_0 j_4$$

Define 4-potential $\boxed{A_\mu = (\vec{A}, iV/c)}$

$$\Rightarrow \quad \Box^2 A_\mu = -\mu_0 j_\mu \quad \text{equation for potentials}$$

$\Box^2 = \dfrac{\partial^2}{\partial x_\nu^2}$ is Lorentz invariant operator

So we can write the above as $\boxed{\dfrac{\partial^2 A_\mu}{\partial x_\nu^2} = -\mu_0 j_\mu}$

Lorentz gauge condition is

$$0 = \vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A} + \frac{\partial (iV/c)}{\partial (ict)} = \vec{\nabla}\cdot\vec{A} + \frac{\partial A_4}{\partial x_4}$$

$$= \frac{\partial A_\mu}{\partial x_\mu}$$

So Lorentz Gauge condition is $\boxed{\dfrac{\partial A_\mu}{\partial x_\mu} = 0}$

Electric and Magnetic Fields

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \Rightarrow \quad \boxed{B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}}$$

where i, j, k are a cyclic permutation of $1, 2, 3$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$V = \frac{cA_4}{i}, \quad x_4 = ict$$

$$\Rightarrow E_i = -\frac{\partial\left(\frac{c}{i} A_4\right)}{\partial x_i} - \frac{\partial A_i}{\partial\left(\frac{x_4}{ic}\right)} = -\frac{c}{i}\frac{\partial A_4}{\partial x_i} - ic\frac{\partial A_i}{\partial x_4}$$

$$\boxed{\frac{E_i}{c} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)}$$

has a similar form to B_i

We define the field strength tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}} = -F_{\nu\mu}$$

4×4 antisymmetric 2nd rank tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4×4 antisymmetric 2nd rank tensor

4×4 antisymmetric 2nd rank tensor has only 6 independent components — just the right number to specify the $\vec{E}$ and $\vec{B}$ fields!