Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{when } ijk \text{ is an even permutation of } 123 \\ -1 & \text{when } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise, in particular if any of the two } ijk \text{ are equal} \end{cases}$$

ijk is an <u>even</u> permutation of 123 if you can get to it by making an <u>even</u> number of pairwise interchanges. An <u>odd</u> permutation requires an <u>odd</u> number of pairwise interchanges.

213 is an ~~even~~ odd permutation $123 \to 213$
one switch $\equiv$ 1 interchange

231 is an ~~odd~~ even permutation $123 \to 213 \to 231$
switch switch $\equiv$ 2 interchanges

~~[illegible]~~

If $\vec{A} = \bar{B} \times \bar{C}$ then

ith component of A is $A_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} B_j C_k$

For example $(1=x,\ 2=y,\ 3=z)$

$$A_1 = \sum_{jk} \varepsilon_{1jk} B_j C_k$$

By properties of ε_{ijk}, the only terms in above sum that are not zero are $(j,k) = (2,3)$ and $(3,2)$

$\varepsilon_{123} = +1, \quad \varepsilon_{132} = -1$

So

$$A_1 = B_2 C_3 - B_3 C_2$$ this is just the x-component of $\bar{B} \times \bar{C}$

$$\sum_{jklm} \left[\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl} \right] A_j B_l C_m$$

$$= \sum_j A_j B_i C_j - \sum_j A_j B_j C_i$$

$$= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B}) \quad \text{as } \vec{A} \cdot \vec{C} = \sum_j A_j C_j$$

so

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B})$$

BAC − CAB rule!

You can check that the other components also come out correct

$A_2 = B_3 C_1 - B_1 C_3 \quad , \quad A_3 = B_1 C_2 - B_2 C_1$

A useful relation is

$$\sum_{i=1}^{3} \varepsilon_{ijk}\, \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

Since $\varepsilon_{ijk} = 0$ unless i, j, k are all different, the above will be non zero only if the pair (j,k) has the same numbers as the pair (l,m), and $j \neq k$ and $l \neq m$. When $j = l$ and $k = m$, then the above is $(\varepsilon_{ijk})^2 = +1$. When $j = m$ and $k = l$, then the above is $\varepsilon_{ijk}\,\varepsilon_{ikj} = -1$.

You can check that the right hand side obeys all these properties

Example: $\vec{A} \times (\vec{B} \times \vec{C})$

i^{th} component of the above is

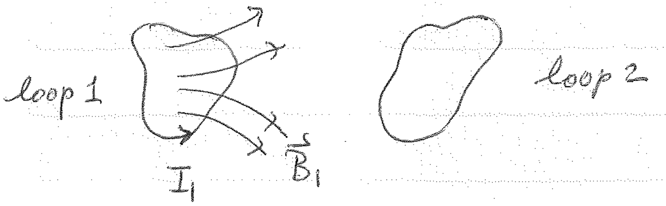
$$\sum_{jklm=1}^{3} \varepsilon_{ijk} A_j \underbrace{\varepsilon_{klm} B_l C_m}_{k^{th} \text{ component of } \vec{B}\times\vec{C}}$$

$$= \sum_{jklm} \varepsilon_{kij}\, \varepsilon_{klm} A_j B_l C_m = \sum_{jklm} \left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] A_j B_l C_m$$

$\varepsilon_{ijk} = \varepsilon_{kij}$ since 2 pair interchanges take from ijk to kij

Inductance

Mutual Inductance



$$= \vec{A}(\vec{r}_2) = \frac{\mu_0}{4\pi}\int d^3r_1' \frac{\vec{J}(\vec{r}_1')}{|\vec{r}_2 - \vec{r}_1|}$$

What is magnetic flux through loop 2, due to current flowing in loop 1?

field produced by $\vec{I}_1$

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{S_2} (\vec{\nabla}\times\vec{A}_1)\cdot d\vec{a}_2 = \oint_{\Gamma_2} \vec{A}_1 \cdot d\vec{l}_2$$

S_2 surface enclosed by loop 2

in Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $\quad \vec{A}_1(\vec{r}_2) = \frac{\mu_0}{4\pi}\oint_{\Gamma_1} \frac{dl_1\, \vec{I}_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$

Γ_1 loop 1

$$\Rightarrow \Phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{\Gamma_2}\left(\oint_{\Gamma_1}\frac{d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}\right)\cdot d\vec{l}_2$$

$$= \frac{\mu_0}{4\pi} I_1 \oint_{\Gamma_1}\oint_{\Gamma_2} \frac{d\vec{l}_1\cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|} \equiv M_{21} I_1 \qquad M_{21} = \frac{\mu_0}{4\pi}\oint\oint\frac{d\vec{l}_1\cdot d\vec{l}_2}{|\vec{r}_2 - \vec{r}_1|}$$

mutual inductance of loops 1 and 2.

Similarly, flux through loop 1, due to current I_2 in loop 2 is:

$$\Phi_1 = \frac{\mu_0}{4\pi} I_2 \oint_{\Gamma_2}\oint_{\Gamma_1} \frac{d\vec{l}_2\cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|} \equiv M_{12} I_2$$

we see that $M_{12} = M_{21}$

$M_{12} = M_{21} \equiv M$ is a purely geometrical quantity.

Flux through loop 2 when I flows in loop 1
= Flux through loop 1 when I flows in loop 2

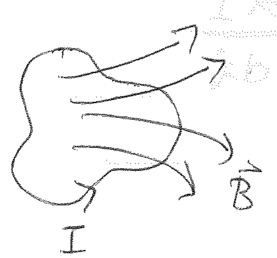
for <u>any</u> two loops.

If vary current in loop 1, flux through loop 2 changes
$\Rightarrow$ emf develops around loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

$\Rightarrow$ induced current $I_2 = \frac{\mathcal{E}_2}{R_2}$ ← resistance of loop 2
flows in loop 2
when current in loop 1 is changed.
This is the principle behind a transformer.

<u>Self Inductance</u>



What is magnetic flux through loop, due to current flowing in loop?

$$\Phi = \oint_\Gamma \vec{A}\cdot d\vec{l} = \frac{\mu_0 I}{4\pi}\oint_\Gamma\oint_\Gamma \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|} \equiv L I$$

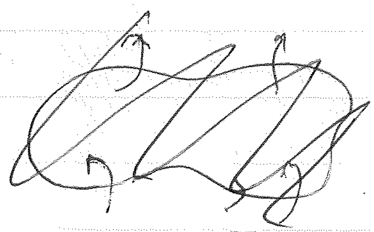
↑ self inductance

↑ both $\vec{r}$ and $\vec{r}'$ lie on same loop Γ.

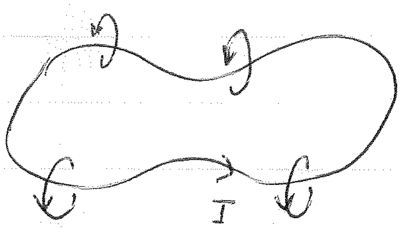
inductance measured in "henries" (H)

1 H = 1 volt-sec/amp

self inductance always positive



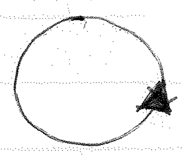
each segment $I \rightarrow$
generates $\vec{B}$ field that circulates around it according to right hand rule



$\Rightarrow$ net flux is always positive for counter clockwise current changing I in loop, changes Φ through loop, creates emf around loop $\mathcal{E} = -\frac{d\Phi}{dt}$

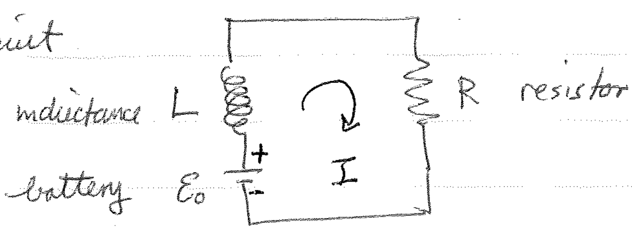
$$\Rightarrow \quad \mathcal{E} = -L\frac{dI}{dt} \qquad\qquad L > 0 \text{ always}$$

this emf $\mathcal{E}$ acts to <u>oppose</u> any change in current — its called the <u>back emf</u>



if I counterclockwise is increased, then $\mathcal{E}$ induced is negative, ie the induced $\mathcal{E}$ tries to drive a current in the opposite (clockwise) direction, to oppose the increase in I

<u>ex:</u> "LR" circuit



total emf in ~~circuit~~ ~~on~~ circuit is:

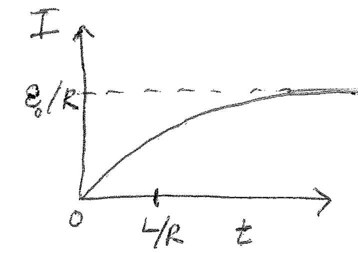
$$\mathcal{E}_0 - L\frac{dI}{dt} = IR$$ ← ohms law for the resistor

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{\mathcal{E}_0}{L}$$

1^{st} order differential equ for $I(t)$.

if switch on battery at $t=0$ } Solution is

$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right)$$



current increases to steady state value $\mathcal{E}_0/R$ over time $t \approx L/R$.