Levi-Civita Tensor

\[ E_{ijk} = \begin{cases} 
+1 & \text{when } ijk \text{ is an even permutation of } 123 \\
-1 & \text{when } ijk \text{ is an odd permutation of } 123 \\
0 & \text{otherwise, i.e., if any of the two } ijk \text{ are equal}
\end{cases} \]

\[ \text{If } ijk \text{ is an even permutation of } 123 \text{ if you can get to it by making an even number of pairwise interchanges.}
\]

An odd permutation requires an odd number of pairwise interchanges.

\[ 213 \text{ is an odd permutation } 123 \rightarrow 213 \]
\[ 1 \text{ switch } \Rightarrow 1 \text{ interchange} \]

\[ 231 \text{ is an even permutation } 123 \rightarrow 213 \rightarrow 231 \]
\[ 2 \text{ switches } \Rightarrow 2 \text{ interchanges} \]

Also consider:

If \( \overline{A} = \overline{B} \times \overline{C} \) then

\[ A_i = \sum_{jk=1}^{3} E_{ijk} B_j C_k \]

For example \((1=x, 2=y, 3=z)\)

\[ A_1 = \sum_{jk} E_{1jk} B_j C_k \]

By properties of \( E_{ijk} \), the only terms in above sum that are not zero are \((j,k) = (2,3) \) and \((3,2)\)

\[ E_{123} = +1, \quad E_{132} = -1 \]

So,

\[ A_1 = B_2 C_3 - B_3 C_2 \quad \text{this is just the } x \text{-component of } \overline{B} \times \overline{C} \]
\[
\sum_{j,k,l,m} \left( \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) A_j B_k C_l
\]

\[
= \sum_j A_j B_j C_j - \sum_i A_i B_j C_i
\]

\[
= B_j \left( \vec{A} \cdot \vec{C} \right) - C_j \left( \vec{A} \cdot \vec{B} \right)
\]

as \( \vec{A} \cdot \vec{C} \)

So

\[
\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})
\]

BAC - CAB rule!
you can check that the other components also come out correct

\[ A_2 = B_3 C_1 - B_1 C_3, \quad A_3 = B_1 C_2 - B_2 C_1. \]

**Useful relation** is

\[ \sum_{c=1}^{3} E_{ijk} E_{ikm} = E_{jel} E_{km} - E_{jml} E_{kme} \]

Since \( E_{ijk} = 0 \) unless \( i,j,k \) are all different.

The above will be non zero only if the pair \((j,k)\) has the same numbers as the pair \((k,m)\).

When \( j = l \) and \( k = m \), then the above is \( (E_{ijk})^2 = +1 \). When \( j = m \) and \( k = l \), then the above is \( E_{ijk} E_{ikj} = -1 \).

You can check that the right hand side obeys all these properties.

**Example:** \( \overline{A} \times (\overline{B} \times \overline{C}) \)

The component of the above is

\[ \sum_{ijklm=1}^{3} E_{ijkl} \overline{A}_{ij} \overline{E}_{klm} \overline{B}_{le} \overline{C}_{lm} \]

The component of \( \overline{B} \times \overline{C} \)

\[ \sum_{jklm} E_{kij} E_{kem} \overline{A}_{ij} \overline{B}_{lme} \overline{C}_{lm} = \sum_{jklm} \left( \overline{E}_{ij} \overline{E}_{im} \overline{A}_{jl} \overline{B}_{le} \overline{C}_{lm} \right) \]

where \( E_{ijk} = E_{kij} \) since 2 pair

interchanges take from \( ijk \) to \( kij \)...
What is magnetic flux through loop 2, due to current flowing in loop 1?

\[ \Phi_2 = \int_{S_2} \mathbf{B} \cdot d\mathbf{A}_2 = \int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{A}_2 = \oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l} \]

in Coulomb gauge, \( \nabla \times \mathbf{A} = 0 \),

\[ \mathbf{A}_1(r_2) = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{I_1 \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} d\mathbf{l}_1 \]

\[ \Rightarrow \Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint_{C_2} \left( \frac{\mathbf{d} \mathbf{l}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \right) \cdot d\mathbf{l}_2 \]

\[ = \frac{\mu_0 I_1}{4\pi} \int_{C_1} \int_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} = M_{21} I_1 \]

\[ M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} \]

is mutual inductance of loops 1 and 2.

Similarly, flux through loop 1, due to current \( I_2 \) in loop 2 is:

\[ \Phi_1 = \frac{\mu_0}{4\pi} I_2 \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_2 \cdot d\mathbf{l}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} = M_{12} I_2 \]

we see that \( M_{12} = M_{21} \)
\[ M_{12} = M_{21} = M \] is a purely geometrical quantity.

Flux through loop 2 when I flows in loop 1
\[ = \text{Flux through loop 1 when I flows in loop 2} \]

for any two loops.

If very current in loop 1, flux through loop 2 changes
\[ \Rightarrow \text{emf develops around loop 2} \]

\[ E_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt} \]

\[ \Rightarrow \text{induced current } I_2 = \frac{E_2}{R_2} \] resistance of loop 2
flows in loop 2
when current in loop 1 is changed.
This is the principle behind a transformer.

\textbf{Self Inductance}

What is magnetic flux through loop, due
to current flowing in loop?

\[ \Phi = \oint \mathbf{A} \cdot d\mathbf{l} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{l} \cdot d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \equiv LI \]

\[ \text{self inductance} \]

both \( \mathbf{r} \) and \( \mathbf{r}' \)
lie on same loop \( \Pi \).

Inductance measured in "henries" (H)

\[ 1 \text{ H} = 1 \text{ volt-sec/amp} \]
Self inductance always positive

Each segment $I \rightarrow$

Generates $B$ field that encircles around it accordingly to right-hand rule

$\phi$ net flux is always positive for counter clockwise current
changing $I$ in loop, changes $\Phi$ through loop, creates emf around loop: $E = -\frac{d\Phi}{dt}$

$\Rightarrow E = -L \frac{dI}{dt}$  \quad $L > 0$ always

This emf $E$ acts to oppose any change in current - it's called the back emf.

If $I$ counter-clockwise is increased, then $E$ induced is negative, i.e. the induced $E$ tries to drive a current in the opposite (clockwise) direction, to oppose the increase in $I$.

**Ex:** "LR" circuit

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\begin{align*}
\text{inductance } L & \quad \text{R resistor} \\
\text{battery } E_0 & \quad \text{I}
\end{align*}
```

Total emf in circuit is:  \quad \text{(Ohm's law for the resistor)}

$$E_0 - L \frac{dI}{dt} = IR$$

if switch on battery at $t=0$

$$\frac{dI}{dt} = -\frac{R}{L} I + \frac{E_0}{L}$$

$1^{st}$ order differential eqn for $I(t)$.

Solution is

$$I(t) = \frac{E_0}{R} \left(1 - e^{-\frac{R}{L}t}\right)$$

current increases to steady state value $\frac{E_0}{R}$ over time $t \approx \frac{L}{R}$. 