Solution: when we constructed the configuration according to method ② we assumed the currents I_1 and I_2 in the two loops stayed constant as the loop 2 was moved into position with respect to loop 1. But as loop 2 moves, the flux through loop 2 due to current in loop 1 changes $\Rightarrow$ emf induced in loop 2. Similarly, flux through loop 1 changes $\Rightarrow$ emf induced in loop 1.
If we want to keep I_1 and I_2 constant there must be some battery in each loop doing work to counter these induced emfs, The work done by these batteries is

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2 \qquad \mathcal{E}_1 = \frac{-d\Phi_1}{dt}$$

$$= I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \qquad \mathcal{E}_2 = -\frac{d\Phi_2}{dt}$$

$$W_{battery} = \int_0^T dt \left(I_1 \frac{d\Phi_1}{dt} + I_2 \frac{d\Phi_2}{dt} \right) = I_1 \Phi_1 + I_2 \Phi_2$$

integrate from $t=0$ when loops infinitely separated to $t=T$ when loops in final position,
As loop moves, I_1 and I_2 stay constant but Φ_1 and Φ_2 change

Φ_1 is flux through loop 1 due to field from loop 2.

$$W_{battery} = I_1 \Phi_1 + I_2 \Phi_2$$

$$= I_1 (M I_2) + I_2 (M I_1)$$

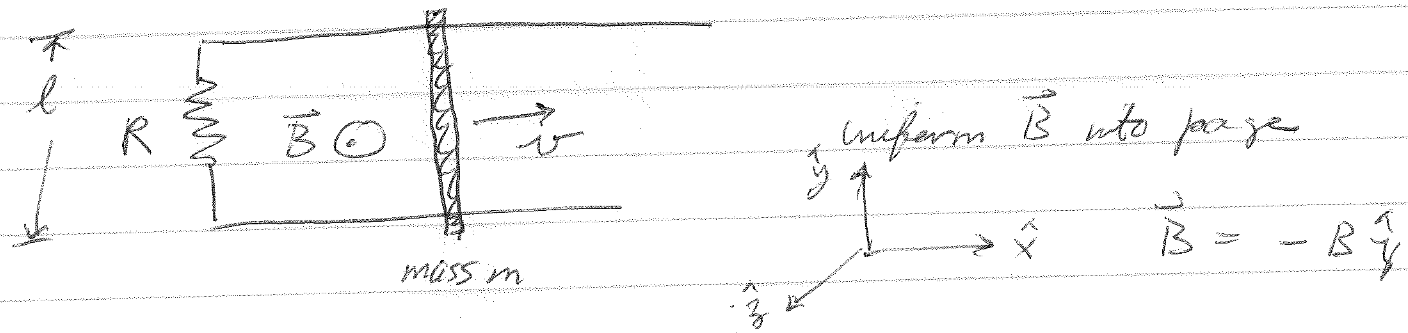
$$= 2 M I_1 I_2$$

Total work is $\tilde{W} + W_{battery} = -M I_1 I_2 + 2 M I_1 I_2 = M I_1 I_2$

Moral: Even in a magnetostatic configuration such as two loops with steady currents, we need to know about dynamics, ie Faraday's law, in order to compute the magnetostatic energy stored in the configuration

7.7

Metal bar of mass m slides on frictionless conducting rails as shown in the diagram



a) If bar moves to right with speed v, what is current in resistor? What direction does it flow.

$\vec{v} = v\hat{x}$

We compute the emf induced in the loop

method ① Lorentz force on electrons in moving bar that drives them ~~through~~ around the loop is

$$\vec{F}_L = q\vec{v}\times\vec{B} = q(v\hat{x})\times(-B\hat{z})$$
$$= qvB\hat{y}$$

$\Rightarrow$ current flows counterclockwise

$$\mathcal{E} = \oint \frac{\vec{F}_L}{q}\cdot d\vec{l} = vBl$$

integrate counterclockwise around loop

current in resistor is $\boxed{I = \frac{\mathcal{E}}{R} = \frac{vBl}{R}}$

method ② By Faraday's law $\Phi = -Bl\,x$

we we compute flux taking normal to loop in $\hat{z}$ direction, and x is distance from sliding bar to the resistor.

Then emf computed counter clockwise around loop is

$$\mathcal{E} = -\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv$$

same as by method ①

b) What is the magnetic force on the bar? In what direction is the force?

Lorentz force on the bar is

$$\vec{F}_{bar} = \int_{bar} I\, d\vec{l} \times \vec{B}$$

$$= I\left(l\hat{y} \times (-B\hat{z})\right)$$

$$\boxed{\vec{F}_{bar} = -IlB\hat{x} = -\frac{vl^2B^2}{R}\hat{x}}$$

$\vec{F}_{bar}$ is directed opposite to direction of motion of the bar

c) If velocity of bar is v_0 at $t=0$, what is $v(t)$ at later times?

Newton's Eqn: $m \frac{d\vec{v}}{dt} = \vec{F}_{bar} = -\frac{v \ell^2 B^2}{R}\hat{x}$

$$\frac{dv}{dt} = -\frac{\ell^2 B^2}{mR} v$$

$$\Rightarrow \boxed{v(t) = v_0 e^{-t/\tau} \quad \text{where } \frac{1}{\tau} = \frac{\ell^2 B^2}{mR}}$$

d) The initial kinetic energy was $\frac{1}{2} m v_0^2$. Show that this is the energy dissipated in the resistor from $t=0$ to $t=\infty$ after bar has stopped moving.

Power dissipated in resistor is $\mathcal{E} I$

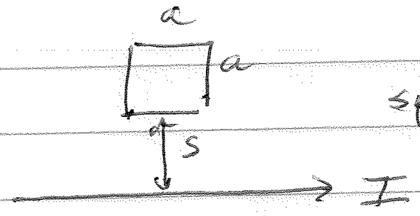
Total energy dissipated is $W = \int_0^\infty dt \; \mathcal{E} I$

$$W = \int_0^\infty dt \, (B\ell v)\left(\frac{vB\ell}{R}\right) = \frac{\ell^2 B^2}{R} \int_0^\infty dt \, v^2(t)$$

$$= \frac{\ell^2 B^2}{R} v_0^2 \int_0^\infty dt \, e^{-2t/\tau} = \frac{\ell^2 B^2 v_0^2}{R} \left(-\frac{\tau}{2}\right) \left[e^{-2t/\tau} \right]_0^\infty$$

$$= \frac{\ell^2 B^2 v_0^2}{R} \frac{\tau}{2} = \frac{\ell^2 B^2 v_0^2}{R} \frac{mR}{2\ell^2 B^2} = \frac{1}{2} m v_0^2$$

7.8 |



square loop distance s from current carrying wire

a) What is flux of magnetic field through the loop?

use cylindrical coordinates with $\vec{I} = I\hat{z}$
magnetic field from the wire is $\vec{B}(\vec{r}) = B(r)\hat{\varphi}$

Ampere: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$
integrate over circle of radius r $\Rightarrow 2\pi r B(r) = \mu_0 I$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$

flux through the loop is $\Phi = a\int_a^{a+s} dr\, \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I a}{2\pi}\ln\left(\frac{a+s}{a}\right)$
(computing flux out of page)

b) If loop is pulled away from wire with speed v, what is the emf around the loop? In what direction will the induced current flow?

$\mathcal{E} = -\frac{d\Phi}{dt}$ Since loop is pulled away from the wire, the flux decreases, so $\mathcal{E} > 0$, so current flows <u>counterclockwise</u>

$$\cancel{\mathcal{E} = \frac{\mu_0 I}{2\pi}\frac{d}{dt}\left(s\ln\left(1+\frac{s}{a}\right)\right) = \frac{\mu_0 I}{2\pi}\left[\ln\left(1+\frac{s}{a}\right) + \frac{s\left(\frac{1}{a}\right)}{1+\frac{s}{a}}\right]}$$

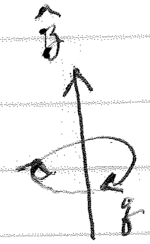
$$\mathcal{E} = \frac{-d\Phi}{dt} = -\frac{\mu_0 I a}{2\pi} \frac{d}{dt}\left[\ln\left(1+\frac{a}{s}\right)\right]$$

$$= -\frac{\mu_0 I a}{2\pi} \frac{\left(\frac{-a}{s^2}\right)\frac{ds}{dt}}{1+a/s} = \frac{\mu_0 I a^2 v}{2\pi\,(s^2+sa)}$$

c) what if loop is pulled to the right at speed v

now $\frac{d\Phi}{dt} = 0$, no current flows

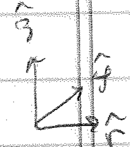
7.50 Betatron: use $\frac{\partial B}{\partial t}$ to accelerate a charge in a circular orbit.



magnetic field $\vec{B} = B(r)\,\hat{z}$ r is cylindrical radial coord

B is cylindrically symmetric about $\hat{z}$

cylindrical coords



~~cyclotron~~ cyclotron motion of charged particle in circular orbit at radius r.

$$m\vec{a} = -\frac{mv^2}{r}\hat{r} = q\,\vec{v}\times\vec{B} = -qvB(r)\,\hat{r}$$

(For $q>0$, $\vec{v}$ must be in $-\hat{\varphi}$ direction, ie charge moves clockwise, in order for Lorentz force to be in $-\hat{r}$ direction)

$$\Rightarrow mv = qrB(r)$$ determines velocity v of charge for circular orbit.

Now suppose $B(r)$ changes with time $\Rightarrow$ magnetic flux through orbit of charge changes $\Rightarrow$ electric field $\vec{E}$ is induced that accelerates the charge.

What condition must hold for the charge to be accelerated, but stay in fixed orbit at radius r?

Let us find the induced $\vec{E}$. For $\vec{B} = B(r)\,\hat{z}$ symmetry gives $\vec{E} = E(r)\,\hat{\varphi}$. $\vec{E}$ depends only on the cylindrical radial distance r, and points in polar direction $\hat{\varphi}$. [Hint: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$. $\frac{\partial \vec{B}}{\partial t}$ is like current flowing down wire, $\vec{E}$ is like resulting magnetic field $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$]

For $\vec{E}(\vec{r}) = E(r)\hat{\varphi}$

$$\vec{\nabla}\times\vec{E} = \frac{1}{r}\frac{d}{dr}\left(rE(r)\right)\hat{z} \quad \text{in cylindrical coords}$$

$$= -\frac{\partial B}{\partial t}\hat{z}$$

$$\Rightarrow \quad \frac{d}{dr}\left(rE(r)\right) = -r\frac{\partial B}{\partial t}$$

$$2\pi\int_0^r dr'\frac{d}{dr'}\left(r'E(r')\right) = -2\pi\int_0^r dr'\, r'\frac{\partial B}{\partial t} = -\frac{\partial \Phi}{\partial t}$$

where Φ is the flux through the orbit of radius r. Define the average magnet field over this orbit, B_{av}, by $\Phi = \pi r^2 B_{av}$.

Integrating left hand side we get

$$2\pi r E(r) = -\pi r^2 \frac{dB_{av}}{dt}$$

$$E(r) = -\frac{r}{2}\frac{dB_{av}}{dt} \qquad \text{for } \frac{dB_{av}}{dt} > 0,\ \vec{E} \propto -\hat{\varphi} \text{ direction}$$

This $\vec{E}$ will accelerate the charge's velocity with which it is going around the orbit. We have for the magnitude of $\vec{v}$

$$m\frac{dv}{dt} = qE = +q\frac{r}{2}\frac{dB_{av}}{dt}$$

charge moves clockwise in $-\hat{\varphi}$ direction

$\vec{E}$ points clockwise in $-\hat{\varphi}$ direction

$q\frac{r}{2}\frac{dB_{av}}{dt}$ gives <u>magnitude</u> of E

$$m\frac{dv}{dt} = q\frac{r}{2}\frac{dB_{av}}{dt}$$

But to maintain circular orbit the cyclotron condition is

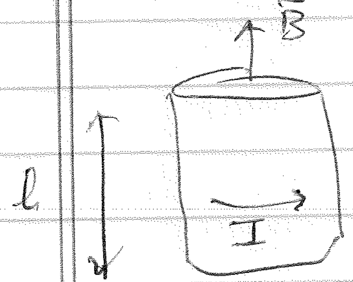
$$mv = qr\,B(r) \Rightarrow m\frac{dv}{dt} = qr\frac{\partial B(r)}{\partial t}$$ if r stays constant

$$\Rightarrow \frac{1}{2}\frac{dB_{av}}{dt} = \frac{\partial B(r)}{\partial t}$$

ie the magnetic field $B(r)$ at the radius of the orbit should be $\frac{1}{2}$ the ~~average~~ magnetic field averaged over the area of the orbit

$$\boxed{B(r) = \frac{1}{2}B_{av}}$$

Self inductance of a solenoid length l, radius R



$$\vec{B} = \mu_0 N I \hat{z}$$

N: # turns of wire per unit length

Total flux through <u>all</u> the wire loops that make up the solenoid is

$$\Phi = \underbrace{(\mu_0 N I \pi R^2)}_{\text{flux through one loop}} \underbrace{(N l)}_{\text{number of loops}} = \mu_0 N^2 l \pi R^2 I$$

$\Phi = L I$ defines self inductance L

$$\Rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$$

Another way to do the calculation:
The energy stored in this magnetostatic configuration is

$$W_{mag} = \frac{1}{2\mu_0} \int d^3r\, |\vec{B}|^2 = \frac{1}{2\mu_0} \underbrace{(\mu_0 N I)^2}_{\text{B-field in solenoid}} \underbrace{(\pi R^2 l)}_{\text{volume inside solenoid}}$$

(we assume only B we need consider is that inside the solenoid)

$$W_{mag} = \frac{1}{2} \mu_0 N^2 l \pi R^2 I^2$$

But we also know $W_{mag} = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \mu_0 N^2 l \pi R^2}$

same result as above