Momentum Conservation

want similar conservation law for mechanical & electromagnetic momentum

$$\frac{\partial}{\partial t}\left(p_{mi} + p_{EBi} \right) = \vec{\nabla} \cdot \vec{T}_i \qquad i = x, y, z$$

p_{mi} = i^{th} component of mechanical momentum density

p_{EBi} = i^{th} component of electromagnetic momentum density

$\vec{T}_i$ = $-\underbrace{\text{flux density}}_{\text{(or "current")}}$ of i^{th} component of momentum density

Since $\vec{T}_i$ is a vector with 3 components, and there are three such vectors, for $i = x, y, z$, we will see that these 3 vectors form the components of a 3X3 tensor (ie matrix)

mechanical momentum density given by Newtons Law $\Big\}$ $\dfrac{\partial \vec{p}_m}{\partial t} = \vec{f} = \rho \vec{E} + \vec{j} \times \vec{B}$

force density $\vec{f} = \rho \vec{E} + \vec{j} \times \vec{B} = \varepsilon_0 (\vec{\nabla} \cdot \vec{E}) \vec{E} + \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \times \vec{B}$

Now apply vector algebra + Maxwell's equ (see text sec 8.2.2) to manipulate into the form

$$\vec{f} = \varepsilon_0 \left[(\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} \right] + \frac{1}{\mu_0} \left[(\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} \right]$$
$$- \frac{1}{2} \vec{\nabla} \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) \qquad \cancel{(7.91)} \ (8.15)$$

$\vec{f}$ above looks like mess, but it simplifies if one introduces the following 3x3 matrix, known as the "Maxwell Stress Tensor"

$$T_{ij} = \varepsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \tfrac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)$$

$$i = x, y, z \quad \text{and} \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\overleftrightarrow{T} = \begin{bmatrix} \varepsilon_0\left(E_x^2 - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_x^2 - \frac{1}{2}B^2\right) & \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \varepsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \\ \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \varepsilon_0\left(E_y^2 - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_y^2 - \frac{1}{2}B^2\right) & \varepsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z \\ \varepsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z & E_y E_z + \frac{1}{\mu_0} B_y B_z & \varepsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) + \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}B^2\right) \end{bmatrix}$$

$T_{ij} = T_{ji} \Rightarrow T$ is symmetric

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T})_j = \sum_i \frac{\partial}{\partial x_i} T_{ij} = \sum_i \Big[\varepsilon_0\left(\frac{\partial E_i}{\partial x_i} E_j + E_i \frac{\partial E_j}{\partial x_i} - \frac{1}{2}\frac{\partial E^2}{\partial x_i}\delta_{ij}\right)$$
$$+ \frac{1}{\mu_0}\left(\frac{\partial B_i}{\partial x_i} B_j + B_i \frac{\partial B_j}{\partial x_i} - \frac{1}{2}\frac{\partial B^2}{\partial x_i}\delta_{ij}\right)\Big]$$

$$(\vec{\nabla} \cdot \overleftrightarrow{T}) = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \tfrac{1}{2}\vec{\nabla}E^2\right]$$
$$+ \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \tfrac{1}{2}\vec{\nabla}B^2\right]$$
$$= \vec{f} + \varepsilon_0 \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{f} + \varepsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t}$$

$$\vec{f} = \frac{\partial \vec{p}_M}{\partial t} = -\varepsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t} + \vec{\nabla}\cdot\overleftrightarrow{T}$$

$$\frac{\partial}{\partial t}\left[\vec{p}_M + \varepsilon_0\mu_0\vec{S}\right] = \vec{\nabla}\cdot\overleftrightarrow{T}$$ this is the desired conservation law for momentum

$\Rightarrow$ $\boxed{\varepsilon_0\mu_0\vec{S} = \vec{p}_{EM}}$ electromagnetic momentum density

~~$-\vec{\nabla}\cdot\overleftrightarrow{T}$ = current or flux~~

$-\overleftrightarrow{T}_{ij}$ is i^{th} component of current, of j^{th} component of electromagnetic momentum density

ie the vector $-\begin{pmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{pmatrix}$ is the current for

x-component $\vec{p}_{EMx}$ of E-M momentum

integral form: $$\int_{Vol} d^3r \left[\frac{\partial}{\partial t}\vec{p}_M + \frac{\partial}{\partial t}\vec{p}_{EB}\right] = \frac{d}{dt}\int_{Vol} d^3r \underbrace{\left(\vec{p}_M + \vec{p}_{EB}\right)}_{\text{total mechanical + electromagnetic field momentum contained in Vol}}$$

$$= \int_{Vol} d^3r\, \vec{\nabla}\cdot\overleftrightarrow{T} = \underbrace{\oint_S d\vec{a}\cdot\overleftrightarrow{T}}_{(-)\text{ flux of field momentum out through surface } S \text{ bounding Vol}}$$

Or we can write

$$\frac{d}{dt}\int_{vol} d^3r\, \vec{p}_{mech} = \frac{d\vec{P}_{mech}}{dt} = -\frac{d}{dt}\int_{vol} d^3r\, \vec{p}_{EB} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$

$$\frac{d\vec{P}_{mech}}{dt} = \underbrace{-\frac{d\vec{P}_{EB}}{dt} + \oint_S d\vec{a}\cdot\overleftrightarrow{T}}_{\text{total electromagnetic force on the volume}}$$

$$\vec{F}_{EB} = \frac{d\vec{P}_{mech}}{dt}$$

For a situation where $\vec{E}$ and $\vec{B}$ are constant in time,

$$\frac{d\vec{P}_{EB}}{dt} = 0$$ and so

$$\frac{d\vec{P}_{mech}}{dt} = \oint_S d\vec{a}\cdot\overleftrightarrow{T}$$ ← gives total electromagnetic force on the volume

this is why $\overleftrightarrow{T}$ is called the Maxwell stress tensor

It is like a pressure acting on the walls of the volume.

Numerically compute $\varepsilon_0\mu_0$, find $\varepsilon_0\mu_0 = \frac{1}{c^2}$ with c = speed of light

$\vec{S} = c^2 \vec{p}_{EB}$

↑ momentum density

↑ energy current

Suppose energy current is made of "particles" that travel with velocity $\vec{c}$. Then $\vec{S} = \vec{c}\, u_{EB}$ u_{EB} is energy density

$u_{EB} = c\, p_{EB}$ = energy-moment relation for photons.

Also can do same for angular momentum

$\vec{L}_{EB} = \vec{r} \times \vec{p}_{EB} = \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B})$

= angular momentum density contained in $\vec{E}$ and $\vec{B}$ fields

see Griffiths Sec 8.2.4

Maxwell's Eqn in potential form

$$\vec{\nabla}\cdot\vec{B}=0 \;\Rightarrow\; \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$ remains true with dynamics

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})$$

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \;\Rightarrow\; \vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$$

$$\boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}}$$

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \;\Rightarrow\; \boxed{\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0} \quad (*)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \;\Rightarrow\; \underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{A})}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}} = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\frac{\partial V}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\Rightarrow \boxed{\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}} \quad (**)$$

Gauge transformations: if $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ then $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} = \vec{B}$

$\vec{A}'$ gives same $\vec{B}$ as $\vec{A}$.

But if we change $\vec{A}\to\vec{A}'$, we also have to change $V\to V'$ so $\vec{E}$ stays same.

~~if $V'=V+a$, const a then $-\vec{\nabla}V'=-\vec{\nabla}V$~~

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V - \frac{\partial\vec{A}'}{\partial t} + \vec{\nabla}\left(\frac{\partial\lambda}{\partial t}\right)$$

$$= -\vec{\nabla}\left(V - \frac{\partial\lambda}{\partial t}\right) - \frac{\partial\vec{A}'}{\partial t}$$

If let $V' = V - \frac{\partial\lambda}{\partial t}$ then

$$\vec{E} = -\vec{\nabla}V' - \frac{\partial\vec{A}'}{\partial t}$$ has same form as before

$\Rightarrow$ the transformation $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, $V' = V - \dfrac{\partial \lambda}{\partial t}$ } called a "gauge" transformation

for any scalar function $\lambda(\vec{r}, t)$, leaves $\vec{B}$ and $\vec{E}$ unchanged, ie:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}'$$
$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V' - \frac{\partial \vec{A}'}{\partial t}$$

We can therefore use this freedom, given by the arbitrary λ, to make $\vec{\nabla}\cdot\vec{A}$ equal to some desired quantity, which will simplify the equations (*)(**) Making such a choice for $\vec{\nabla}\cdot\vec{A}$ is called "fixing" the gauge.

i) <u>Coulomb gauge</u>: same as used in magnetostatics.

Choose $\vec{\nabla}\cdot\vec{A} = 0$

(if had some $\vec{A}'$ such that $\vec{\nabla}\times\vec{A}' = \vec{B}$, but $\vec{\nabla}\cdot\vec{A}' \neq 0$, ~~then~~ we could always find a $\lambda(\vec{r}, t)$ such that $\vec{A} = \vec{A}' + \vec{\nabla}\lambda$ does satisfy $\vec{\nabla}\cdot\vec{A} = 0$) see Griffiths sec 5.4.1

(*) $\Rightarrow$ $\nabla^2 V = -\rho/\varepsilon_0$

solution is $V(\vec{r}, t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3r' \, \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$ } same as in electrostatics

but unlike statics, need also to know $\vec{A}$ in order to get $\vec{E}$.