

Solutions to the wave equation $\square^2 f = 0$.

plane waves: if $g(\phi)$ is any function of single variable ϕ ,

then

$f(\vec{r}, t) = g(\vec{k} \cdot \vec{r} - \omega t)$ solves wave equation, for any vector \vec{k} ,
and for associated $\omega^2 = v^2 k^2$

proof:

$$\square^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial \phi} \frac{\partial \phi}{\partial x} = \frac{\partial g}{\partial \phi} k_x \quad \text{chain rule}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial \phi} \frac{\partial \phi}{\partial x} \right) = \frac{\partial^2 g}{\partial \phi^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 g}{\partial \phi^2} \left(\frac{\partial \phi}{\partial x} \right)^2 = \frac{\partial^2 g}{\partial \phi^2} k_x^2$$

similarly $\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial \phi^2} k_y^2$, $\frac{\partial^2 g}{\partial z^2} = \frac{\partial^2 g}{\partial \phi^2} k_z^2$, $\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial \phi^2} \omega^2$

$$\square^2 f = \left(k^2 - \frac{\omega^2}{v^2} \right) \frac{\partial^2 g}{\partial \phi^2} = 0 \quad \text{if } \omega^2 = v^2 k^2. \quad \text{for any function } g(\phi)$$

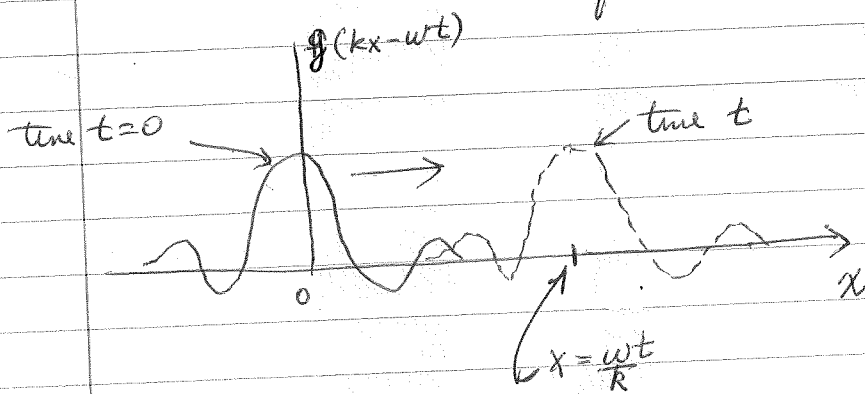
ω is frequency of the wave, i.e. $\lambda = \frac{\omega}{2\pi}$, $T = \frac{1}{\omega}$ is period

$$\begin{aligned} f(\vec{r} + \Delta \vec{r}, t + \Delta t) &= f(\vec{k} \cdot \vec{r} - \omega(t + \Delta t)) \\ &= f(\vec{k} \cdot \vec{r} - \omega t - \omega \Delta t) \end{aligned}$$

plane wave because $f(\vec{r}, t)$ is constant on all planes \perp to \vec{k}
i.e. if $\Delta \vec{r}$ is \perp to \vec{k} , then

$$\begin{aligned} f(\vec{r} + \Delta \vec{r}, t) &= g(\vec{k} \cdot \vec{r} + \underbrace{\vec{k} \cdot \Delta \vec{r}}_{=0} - \omega t) = g(\vec{k} \cdot \vec{r} - \omega t) \\ &= f(\vec{r}, t) \end{aligned}$$

v is ~~velocity~~ speed of wave, suppose $\vec{k} = k \hat{x}$



$g(\phi)$ is called the envelope of the wave

curve shifted to right a distance $x = \frac{\omega}{k} t$ in time t .
 \Rightarrow curve moves with velocity $\vec{v} = \frac{\omega}{k} \hat{x}$

Spherical waves

solution to wave eqn that depends only on radial coord r , i.e. $f(r, t) \leftarrow f$ const on spheres of radius r

$$\left[\nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] f(r, t) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$

try solution of form

$$f(r, t) = \frac{g(kr - \omega t)}{r}$$

call $\phi = kr - \omega t$

$$r^2 \left(\frac{\partial f}{\partial r} \right) = r^2 \left(\frac{1}{r} \frac{dg}{d\phi} k - \frac{g}{r^2} \right) = r \frac{dg}{d\phi} k - g$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) = \frac{1}{r^2} \left[\frac{dg}{d\phi} k + r \frac{d^2 g}{d\phi^2} k^2 - \frac{dg}{d\phi} k \right] = \frac{d^2 g}{d\phi^2} \frac{k^2}{r}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{r} \frac{d^2 g}{d\phi^2} \omega^2$$

so $\frac{g(kr - \omega t)}{r}$ solves wave eqn provided $\frac{d^2 g}{d\phi^2} \frac{k^2}{r} = \frac{1}{r} \frac{d^2 g}{d\phi^2} \frac{\omega^2}{v^2}$

$$\text{i.e. } \omega^2 = v^2 k^2$$

Note: if $f_1(\vec{r}, t)$ and $f_2(\vec{r}, t)$ are solutions to wave eqn, then so is $f_1 + f_2$, as \square^2 is a linear operator

Back to plane waves

A particular solution: sinusoidal wave

$$f(\vec{r}, t) = A \cos(\vec{k} \cdot \vec{r} - \omega t + \delta)$$

period of wave is $T = \frac{2\pi}{\omega}$: $f(\vec{r}, t+T) = f(\vec{r}, t)$

frequency of wave is $\nu = \omega/2\pi = 1/T$

angular freq is ω

wavelength is $\lambda = \frac{2\pi}{|\vec{k}|}$: $f(\vec{r} + \lambda \hat{k}, t) = f(\vec{r}, t)$

wave vector is \vec{k} wavenumber is $k = |\vec{k}|$

amplitude is A

phase constant is δ

phase velocity is $\vec{v} = \frac{\omega}{|\vec{k}|} \hat{k}$ travels in direction \hat{k}

ex: if $\vec{k} = k\hat{x}$ then wave travels in $+\hat{x}$ direction

if $\vec{k} = -k\hat{x}$ then $A \cos(-kx - \omega t + \delta)$ travels in $-\hat{x}$ direction

Complex notation $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(x, t) = \text{Re} \left[A e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} \right]$$

usually we will not write the "Re" ~~by~~ but leave it implied.

$$f(x, t) = \text{Re} \left[\underbrace{A e^{i\delta}}_{\text{complex amplitude}} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]$$

Sinusoidal waves particularly important, since we can expand any solution of wave eqn in terms of them
 \Rightarrow theory of Fourier Transforms

Fourier Series: Any function $f(x)$ defined on $[-\frac{L}{2}, \frac{L}{2}]$ can be expressed as

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi n x}{L}\right) + B_n \sin\left(\frac{2\pi n x}{L}\right) \right]$$

where $A_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) \cos\left(\frac{2\pi n x}{L}\right)$

$$B_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

rewrite in terms of complex exponential.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ \left(\frac{A_n}{2} + \frac{B_n}{2i}\right) e^{i\frac{2\pi n x}{L}} + \left(\frac{A_n}{2} - \frac{B_n}{2i}\right) e^{-i\frac{2\pi n x}{L}} \right\}$$

where

$$\begin{aligned} \frac{A_n}{2} \pm \frac{B_n}{2i} &= \frac{A_n \mp i B_n}{2} = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) \frac{1}{2} \left\{ \cos\frac{2\pi n x}{L} \mp i \sin\frac{2\pi n x}{L} \right\} \\ &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{\mp i\frac{2\pi n x}{L}} \end{aligned}$$

$n = 1, 2, \dots$

$$\text{Define } f_n \equiv \frac{A_n}{2} + \frac{B_n}{2i} = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i \frac{2\pi n x}{L}}$$

$$f_{-n} \equiv \frac{A_n}{2} - \frac{B_n}{2i} = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i \frac{2\pi(-n)x}{L}}$$

$$f_0 \equiv \frac{A_0}{2} = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x)$$

$$f(x) = f_0 + \sum_{n=1}^{\infty} \left\{ f_n e^{i \frac{2\pi n x}{L}} + f_{-n} e^{i \frac{2\pi(-n)x}{L}} \right\}$$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i \frac{2\pi n x}{L}}, \quad f_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i \frac{2\pi n x}{L}}$$

Fourier series in complex form

Now let $L \rightarrow \infty$

$$\text{Define } k_n = \frac{2\pi n}{L} \rightarrow k_{n+1} - k_n = \Delta k = \frac{2\pi}{L}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta k}{2\pi} L f_n e^{i k_n x}$$

$$\text{Define } \tilde{f}(k_n) \equiv \frac{L}{2\pi} f_n \equiv \frac{1}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx f(x) e^{-i k_n x}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \Delta k \tilde{f}(k_n) e^{i k_n x}$$

as $L \rightarrow \infty$, $\Delta k \rightarrow 0$, $\sum \Delta k \rightarrow \int dk$

$$f(x) = \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{i k x}, \quad \tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) e^{-i k x}$$

Fourier transform

$\tilde{f}(k)$ is the Fourier transform of $f(x)$.

$$f = f^*$$

If $f(x)$ is a real function, then

$$\begin{aligned} \tilde{f}(-k) &= \int_{-\infty}^{\infty} dx f(x) e^{+ikx} = \int_{-\infty}^{\infty} dx f(x) (e^{-ikx})^* \\ &= \left(\int_{-\infty}^{\infty} dx f(x) e^{-ikx} \right)^* = \tilde{f}^*(k) \end{aligned}$$

$$\boxed{\tilde{f}(k) = \tilde{f}^*(k)}$$

For a function of 3-dim space,

$$f(x, y, z) = \int_{-\infty}^{\infty} dk_x dk_y dk_z \tilde{f}(k_x, k_y, k_z) e^{ik_x x} e^{ik_y y} e^{ik_z z}$$

$$\text{or } f(\vec{r}) = \int_{-\infty}^{\infty} d^3k \tilde{f}(\vec{k}) e^{i\vec{k} \cdot \vec{r}}$$

$$\text{where } \tilde{f}(\vec{k}) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} f(\vec{r}) e^{-i\vec{k} \cdot \vec{r}}$$

For function of \vec{r} and t

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

where

$$\tilde{f}(\vec{k}, \omega) = \int_{-\infty}^{\infty} \frac{d^3r}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dt}{2\pi} f(\vec{r}, t) e^{-i(\vec{k} \cdot \vec{r} - \omega t)}$$

Fourier transforms are of enormous help in solving partial differential equations.