

General solution to wave equation

$$\square^2 f(\vec{r}, t) = 0 \quad \text{subst in F.T.}$$

$$\square^2 \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) \left[ \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\nabla^2 e^{i\vec{k} \cdot \vec{r}} = \vec{\nabla} \cdot \left( \vec{\nabla} e^{i\vec{k} \cdot \vec{r}} \right) = \vec{\nabla} \cdot \begin{pmatrix} \frac{\partial}{\partial x} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial y} e^{i(k_x x + k_y y + k_z z)} \\ \frac{\partial}{\partial z} e^{i(k_x x + k_y y + k_z z)} \end{pmatrix}$$

$$= \vec{\nabla} \cdot \begin{pmatrix} ik_x \\ ik_y \\ ik_z \end{pmatrix} e^{i\vec{k} \cdot \vec{r}} = \vec{\nabla} \cdot (i\vec{k} e^{i\vec{k} \cdot \vec{r}})$$

product rule:

$$= i\vec{k} \cdot \vec{\nabla} e^{i\vec{k} \cdot \vec{r}} = (i\vec{k}) \cdot (i\vec{k}) e^{i\vec{k} \cdot \vec{r}} = -k^2 e^{i\vec{k} \cdot \vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\Rightarrow \left[ \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \left( -k^2 + \frac{\omega^2}{v^2} \right) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

So wave equ is

$$\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) \left( -k^2 + \frac{\omega^2}{v^2} \right) e^{i(\vec{k} \cdot \vec{r} - \omega t)} = 0$$

$$\Rightarrow \tilde{f}(\vec{k}, \omega) \left( -k^2 + \frac{\omega^2}{v^2} \right) = 0 \quad \therefore \text{if a function} = 0, \text{ then its F.T.} = 0 \quad (\text{all Fourier coefficients} = 0)$$

either

$\Rightarrow \tilde{f}(\vec{k}, \omega) = 0$ , i.e. there is no " $\vec{k}$ ," component in solution

or  $k^2 v^2 = \omega^2$ .  $\therefore$  for each  $\vec{k}$ , only  $\omega = v|\vec{k}|$  is present

So most general solution is

$$f(\vec{r}, t) = \int_{\infty}^{\infty} d^3k \quad \tilde{f}(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\text{where } \omega^2 = v^2 k^2$$

[ $\omega$  is not arbitrary + integrated over, only  $\omega = v|\vec{k}|$  enters] and  $\tilde{f}(\vec{k})$  is anything

General solution is ~~sum~~ linear combination of sinusoidal waves.

In most problems therefore, it will be enough to determine how plane <sup>sinusoidal</sup> waves with wavevector  $\vec{k}$  behave. The general solution can then always be represented as a linear combination of these plane sinusoidal waves.

Alternatively, if  $f(\vec{r}, t=0) = f_0(\vec{r})$  given function, then this will evolve in time according to

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \quad \tilde{f}_0(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \omega^2 = v^2 k^2$$

$$\text{where } \tilde{f}_0(\vec{k}) = \int \frac{d^3r}{(2\pi)^3} f_0(\vec{r}) e^{-i\vec{k} \cdot \vec{r}}$$

Inhomogeneous wave equation

$$\square^2 f(\vec{r}, t) = g(\vec{r}, t) \quad \text{where } g \text{ is a given source function}$$

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$g(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{g}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

substitute in

$$\Rightarrow \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{f}(\vec{k}, \omega) \left[ -k^2 + \frac{\omega^2}{v^2} \right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \tilde{g}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

equate Fourier components

$$\Rightarrow \tilde{f}(\vec{k}, \omega) \left[ -k^2 + \frac{\omega^2}{v^2} \right] = \tilde{g}(\vec{k}, \omega)$$

$$\Rightarrow \tilde{f}(\vec{k}, \omega) = \frac{\tilde{g}(\vec{k}, \omega)}{\frac{\omega^2}{v^2} - k^2}$$

$$\vec{f}(\vec{r}, t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{\tilde{g}(\vec{k}, \omega)}{\frac{\omega^2}{v^2} - k^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}{\frac{\omega^2}{v^2} - k^2} \cdot \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' g(\vec{r}', t') e^{-i(\vec{k}\cdot\vec{r}' - \omega t')}$$

$$f(\vec{r}, t) = \int d^3k \int d\omega \frac{1}{(2\pi)^4} \frac{\omega^2}{v^2 - k^2} \int d^3r' e^{i\vec{k}\cdot\vec{r}' - i\omega t}$$

$$= \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' g(\vec{r}', t')$$

$$\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega \frac{1}{(2\pi)^4} e^{i\vec{k}\cdot(\vec{r}-\vec{r}') - i\omega(t-t')} \frac{e}{\left(\frac{\omega^2}{v^2} - k^2\right)}$$

Green's function for wave eqn

$$G(\vec{r}-\vec{r}', t-t')$$

$$f(\vec{r}, t) = \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt' G(\vec{r}-\vec{r}', t-t') g(\vec{r}', t')$$

## Polarization

vector wave  $\vec{f}(\vec{r}, t) = \vec{A} e^{-i(\vec{k} \cdot \vec{r} - \omega t)}$

where  $\vec{A} = A \hat{m}$ ,  ~~$A \hat{m}$~~ ,  $A$  may be complex number to include phase factor.

if  $\hat{m} \parallel \hat{k}$  we have longitudinal polarization  
 "  $\hat{m} \perp \hat{k}$  " " transverse polarization

## Plane EM waves in vacuum

$$\vec{\nabla} \cdot \vec{E} = \rho = 0 \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Circular polarization: sum of orthogonal transverse polarizations,  $\pi/2$  out of phase

Consider  $\vec{f}(\vec{r}, t) = A \hat{m}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t)} + i A \hat{m}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$   
 $A \hat{m}_2 e^{i(\vec{k} \cdot \vec{r} - \omega t + \pi/2)}$

where  $\hat{m}_1 \perp \hat{m}_2 \perp \hat{k}$

$$\vec{f} = A (\hat{m}_1 + i \hat{m}_2) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

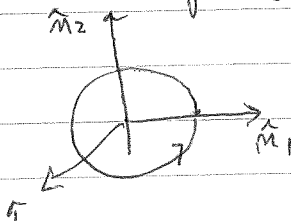
to get physical field, take Real part of complex form

$$\Rightarrow \vec{f} = A \hat{m}_1 \cos(\vec{k} \cdot \vec{r} - \omega t) - A \hat{m}_2 \sin(\vec{k} \cdot \vec{r} - \omega t)$$

$$= A \hat{m}_1 \cos(\omega t - \vec{k} \cdot \vec{r}) + A \hat{m}_2 \sin(\omega t - \vec{k} \cdot \vec{r})$$

$$|\vec{f}|^2 = A^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t) + A^2 \sin^2(\vec{k} \cdot \vec{r} - \omega t) = A^2$$

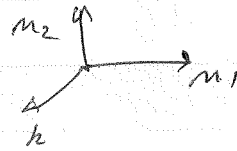
amplitude constant ~~in time~~; but direction of  $\vec{f}$  rotates in time counter clockwise with frequency  $\omega$ .



$$\vec{E} = A(\hat{m}_1 + i\hat{m}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

is a "Right handed" circularly polarized wave

, where  $\hat{m}_1, \hat{m}_2, \hat{k}$  form right handed coord system



$$\vec{E} = A(\hat{m}_1 - i\hat{m}_2) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

is a "Left handed" circularly polarized wave - direction of  $\vec{E}$  rotates in time clockwise ~~with~~