Plane EM waves in a vacuum

\[
\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\n\n\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}
\]

Assume solutions of form \[ \begin{aligned}
\vec{E} &= \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\
\vec{B} &= \vec{B}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)}
\end{aligned} \]

\[ \omega = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c k \]

1) \( i \vec{k} \cdot \vec{E}_0 = 0 \)
2) \( i \vec{k} \cdot \vec{B}_0 = 0 \)
3) \( i \vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \)
4) \( i \vec{k} \times \vec{B}_0 = \mu_0 \varepsilon_0 (-i \omega) \vec{E}_0 \)

(1) and (3) \( \Rightarrow \) EM waves are transverse polarized. \( \vec{E}_0 \) and \( \vec{B}_0 \), both \( \perp \) to \( \vec{k} \).

2) \( \vec{B}_0 = \frac{k}{\omega} \vec{k} \times \vec{E}_0 = \frac{1}{c} \vec{k} \times \vec{E}_0 \Rightarrow \vec{B}_0 \perp \vec{E}_0 \)

\[ |\vec{B}_0| = \frac{1}{c} |\vec{E}_0| \]

\[ \equiv \text{very important factor } \frac{1}{c}! \]

Since Lorentz force is \( \vec{F} = q (\vec{E} + \vec{v} \times \vec{B}) \), the force on a charged particle due to an electromagnetic wave is predominantly from the electric field \( \vec{E} \). The force due to the magnetic field \( \vec{v} \times \vec{B}_0 = \left( \frac{v}{c} \right) \vec{E}_0 \) is reduced by a factor \( \left( \frac{v}{c} \right) \ll 1 \) unless charge is moving relativistically fast.
Energy + momentum in EM wave:

\[ E(\mathbf{r},t) = E_0 \cos(k_y y - wt) \mathbf{\hat{x}} \]

\[ B(\mathbf{r},t) = \frac{1}{2} \mu_0 E_0 \cos(k_y y - wt) \mathbf{\hat{y}} \]

Energy density

\[ U_{EB} = \frac{E_0^2}{2} + \frac{1}{2} \mu_0 B^2 = \frac{E_0^2}{2} \cos^2(k_y y - wt) + \frac{1}{2 \mu_0 c^2} E_0^2 \cos^2(k_y y - wt) \]

\[ = \frac{1}{2} E_0^2 \cos^2(k_y y - wt) \left[ \frac{1}{\varepsilon_0} + \frac{\mu_0}{\varepsilon_0} \right] \]

\[ \leq \frac{1}{\mu_0 \varepsilon_0} \frac{E_0^2}{\varepsilon_0} \]

\[ U_{EB} = E_0^2 \cos^2(k_y y - wt) \]

Energy current

\[ \mathbf{\hat{s}} = \frac{1}{\mu_0} (\mathbf{\hat{E}} \times \mathbf{\hat{B}}) \]

\[ = \frac{1}{\mu_0 c} E_0^2 \cos^2(k_y y - wt) (\mathbf{\hat{x}} \times \mathbf{\hat{y}}) = c E_0^2 \cos^2(k_y y - wt) \mathbf{\hat{y}} \]

\[ \mathbf{\hat{s}} = c U_{EB} \mathbf{\hat{y}} \]

Momentum density

\[ \mathbf{\mathbf{\hat{p}_{EB}}} = \frac{1}{c^2} \mathbf{\hat{s}} = \frac{U_{EB}}{c} \mathbf{\hat{s}} \]

\[ \Rightarrow U_{EB} = c |\mathbf{\mathbf{\hat{p}_{EB}}}| \quad \text{- energy-momentum relation of photons} \]

Since for visible light \( A \sim 5 \times 10^{-7} \text{ m} \sim 5000 \text{ Å} \)

\[ T = \frac{A}{c} = \frac{5 \times 10^{-7}}{3 \times 10^8} \text{ sec} = 1.6 \times 10^{-15} \text{ sec} \]

For most classical measurements, on macroscopic scale,
the measurement will average over many oscillations of the wave. Therefore one is interested in averages

\[ \langle U_{eb} \rangle = \frac{1}{T} \int_{0}^{T} U_{eb} \, dt \]

average over one period of oscillation

\[ = \frac{1}{2} \frac{E_0 E_0^2}{c} \int_{0}^{T} \cos^2(kz - \omega t) \, dt \]

average of \( \cos^2(\phi) \) over one period is \( \frac{1}{2} \)

\[ \langle U_{eb} \rangle = \frac{1}{2} E_0 E_0^2 \]

\[ \langle S \rangle = c \langle U_{eb} \rangle \]

\[ \langle P_{eb} \rangle = \frac{1}{c} \langle U_{eb} \rangle \]

\[ \text{intensity} = \text{average power per area transmitted by wave} \]

\[ I = \langle S \rangle \hat{n} \]

\[ \text{normal to surface through which energy transmuted} \]

\[ I = |\langle S \rangle| \]

\[ \text{magnitude of energy current} \]

\[ \sim \text{amplitude of field}^2 \]

\[ \langle S \rangle \cdot \hat{n} = \text{average power per area transmitted through surface with normal } \hat{n} \]
Maxwell's Equations

\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{tot}} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]

want to write \( \mathbf{J}_{\text{tot}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{b}} \)

in statics: \( \mathbf{J}_{\text{b}} = -\nabla \cdot \mathbf{P} \)

\( \mathbf{J}_{\text{b}} = \nabla \times \mathbf{M} \)

in dynamics: conservation of bound charge \( \Rightarrow \nabla \cdot \mathbf{J}_{\text{b}} = -\frac{\partial \rho_{\text{b}}}{\partial t} \)

\[ \nabla \cdot (\nabla \times \mathbf{M}) = +\frac{\partial}{\partial t} (\nabla \cdot \mathbf{P}) \]

\[ \Rightarrow \frac{\partial}{\partial t} (\nabla \cdot \mathbf{P}) \]

something must be missing! The bound charge arising from \( \mathbf{M} \) must not be all the bound current. There must be bound current arising from a time-varying \( \mathbf{P} \).

bound current from polarization, \( \mathbf{J}_p \) must satisfy

\[ \nabla \cdot \mathbf{J}_p = -\frac{\partial \rho_{\text{b}}}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \mathbf{P}}{\partial t} = \nabla \cdot \left( \frac{\partial \mathbf{P}}{\partial t} \right) \]

\[ \Rightarrow \mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} \]

\[ \Rightarrow \mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \]
\[ \vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} (\vec{P} - \vec{\nabla} \cdot \vec{P}) \]  
\[ \vec{\nabla} \cdot \vec{B} = 0 \]

\[ \vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]
\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Define \[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \]
\[ \vec{H} = \frac{\mu_0}{\sigma} \vec{B} - \vec{M} \]

\[ \Rightarrow \quad \vec{\nabla} \cdot \vec{D} = \rho \]
\[ \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

Inhomogeneous \quad \text{field}

\[ \vec{\nabla} \cdot \vec{B} = 0 \]
\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Homogeneous \quad \text{field}

For linear materials, \[ \vec{D} = \varepsilon \vec{E} \]
\[ \vec{H} = \frac{\mu_0}{\mu} \vec{B} \]  
\{ closes \quad above \quad equations. \}
If we had \( \mathbf{D}(r,t) = \varepsilon \mathbf{E}(r,t) \)
\[ \mathbf{H}(r,t) = \frac{1}{\mu} \mathbf{B}(r,t) \]

then Maxwell's equations, in absence of free charge and free current, would be

\[
\begin{align*}
\varepsilon \nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}
\end{align*}
\]

Everything would be the same except \( \varepsilon_0 \mu_0 \rightarrow \varepsilon \mu > \varepsilon_0 \mu_0 \).

The speed of EM waves in the material would be

\[
\nu = \frac{1}{\sqrt{\varepsilon \mu}} < c \quad \text{index of refraction}
\]

would have \( |\mathbf{B}| = \frac{1}{\mu} |\mathbf{E}| \)

In general however, things are much more complicated for time varying response.

Consider model for polarization of a neutral atom, that we saw last semester.

If displace center of electron cloud from origin by distance \( R \), then there is a restoring force

\[
\mathbf{F}_{\text{rest}} = -\frac{e^2}{4\pi\varepsilon_0 R^3} \mathbf{r} = -m_0 \omega_0^2 \mathbf{r}
\]

(electric field from electron cloud increases linearly with distance from origin)