

If we had $\vec{D}(\vec{r}, t) = \epsilon E(\vec{r}, t)$
 $\vec{H}(\vec{r}, t) = \frac{1}{\mu} B(\vec{r}, t)$

then Maxwell's eqns, in absence of free charge + free current would be

$$\begin{aligned} \epsilon \vec{\nabla} \cdot \vec{E} &= 0 & \vec{\nabla} \times \vec{B} &= \mu \epsilon \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

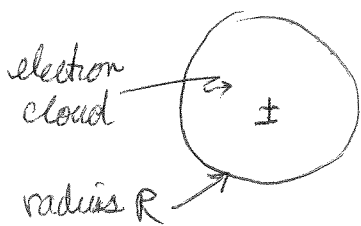
everything would be the same except $\epsilon_0 \mu_0 \rightarrow \epsilon \mu > \epsilon_0 \mu_0$
 The speed of EM waves in the material would be

$$v = \frac{1}{\sqrt{\epsilon \mu}} < c \quad c/v \equiv n \text{ index of refraction}$$

would have $|\vec{B}| = \frac{1}{v} |\vec{E}|$

In general however, things are much more complicated for time varying response

Consider model for polarization of a neutral atom, that we saw last semester



If displace center of electron cloud from ion by distance \vec{r} , then there is a restoring force

$$\vec{F}_{\text{rest}} = \frac{-e^2 \vec{r}}{4\pi \epsilon_0 R^3} \equiv -m \omega_0^2 \vec{r}$$

(electric field from electron cloud increases linearly with distance from origin)

ω_0 has units of frequency.
 ↑
 electron mass

Also in general will be a damping, or friction force, due to energy transfer from atom to other degrees of freedom

$$\vec{F}_{\text{damp}} = -m\gamma \frac{d\vec{r}}{dt} \quad \text{friction} \sim \text{velocity}$$

If electron is in external electric field, the eqn of motion is then

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_{\text{tot}} = -e\vec{E}(t) - m\omega_0^2 \vec{r} - m\gamma \frac{d\vec{r}}{dt}$$

or $\frac{d^2 \vec{r}}{dt^2} + \gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = -\frac{e\vec{E}(t)}{m}$ (assuming that \vec{E} is constant over spatial distances that electron moves)

driven damped harmonic oscillator!

Consider sinusoidal \vec{E} field, in complex form,

$$\vec{E}(t) = \vec{E}_\omega e^{-i\omega t}$$

Assume solution $\vec{r}(t) = \vec{r}_\omega e^{-i\omega t}$

$$-\omega^2 \vec{r}_\omega - i\omega\gamma \vec{r}_\omega + \omega_0^2 \vec{r}_\omega = -\frac{e\vec{E}_\omega}{m}$$

$$\vec{r}_\omega = \frac{-e}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_\omega$$

dipole moment $\vec{p}(t) = -e\vec{r}(t) = \vec{p}_\omega e^{-i\omega t}$

$$\vec{p}_\omega = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_\omega$$

↑ resonance at $\omega \approx \omega_0$
width of resonance is γ

frequency dependent polarizability

$$\vec{P}_\omega = \alpha(\omega) \vec{E}_\omega$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

varies with ω

Since $\alpha(\omega)$ is complex the polarization does not in general oscillate in phase with the electric field.

For a pure harmonic, $\vec{P}(t) = \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$ phase shift from \vec{E}

$$= |\alpha(\omega)| \vec{E}_\omega e^{-i(\omega t - \delta)}$$

where δ is the phase of complex α
 i.e. $\alpha = |\alpha| e^{i\delta}$

For a general electric field $\vec{E}(t) = \int_{-\infty}^{\infty} d\omega \vec{E}_\omega e^{-i\omega t}$

the response is $\vec{P}(t) = \int_{-\infty}^{\infty} d\omega \vec{P}_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$

substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') e^{i\omega t'}$ to get

$$\vec{P}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \int_{-\infty}^{\infty} d\omega \alpha(\omega) e^{-i\omega(t-t')}$$

define Fourier transform $\tilde{\alpha}(t) \equiv \int_{-\infty}^{\infty} d\omega \alpha(\omega) e^{-i\omega t}$

$$\vec{P}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \tilde{\alpha}(t-t')$$

\vec{P} at t is due to \vec{E} at all other times t' , not only at time t

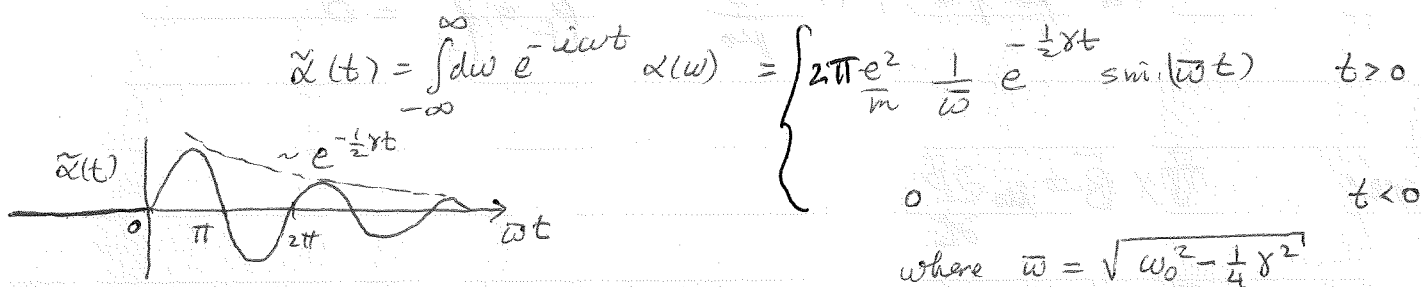
True in general: if $\tilde{A}(\omega) = \tilde{\alpha}(\omega) \tilde{B}(\omega)$ is relation between Fourier transforms, then in time,

$$A(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} B(t') a(t-t')$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \tilde{\alpha}(t-t') \quad \text{response is non-local in time}$$

i.e. $\vec{p}(t)$ is determined not just by the instantaneous $\vec{E}(t)$, but by $\vec{E}(t')$ at other times $t' \neq t$.

For our simple model, $\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$



$\tilde{\alpha}(t) = 0$ for $t < 0 \Rightarrow$ causal response, i.e. $\vec{p}(t)$ depends on $\vec{E}(t')$ only for earlier times $t' < t$

$\tilde{\alpha}(t)$ gives the polarization that results from a δ -function pulse in \vec{E} at time $t'=0$, i.e. $\vec{E}(t') = \vec{E}_0 \delta(t')$

$\tilde{\alpha}(t)$ has the familiar form of the displacement of a damped harmonic oscillator that is given an impulse kick at $t'=0$

For a dielectric, we now expect polarization density from a pure sinusoidal electric field, will be

$$\vec{P}(t) = \vec{P}_\omega e^{-i\omega t} \quad \text{with} \quad \vec{P}_\omega = \epsilon_0 \chi_e(\omega) \vec{E}_\omega$$

$\chi_e(\omega)$ freq dependent electric susceptibility

where $\chi_e(\omega) \approx \frac{N\alpha(\omega)}{\epsilon_0}$
 $N =$ atomic density
 i.e. atoms per volume

for a dilute density of atoms

⇒ Displacement $\vec{D}(t) = \vec{D}_\omega e^{-i\omega t}$ with $\vec{D}_\omega = \epsilon_0 \vec{E}_\omega + \vec{P}_\omega$
 $= \epsilon_0 (1 + \chi_e(\omega)) \vec{E}_\omega$
 $= \epsilon(\omega) \vec{E}_\omega$

$\epsilon(\omega) = \epsilon_0 (1 + \chi_e(\omega))$
 $= \epsilon_0 K(\omega)$

complex, freq dependent permearivity

freq dependent electric susceptibility $\chi_e(\omega) \approx \frac{N\alpha(\omega)}{\epsilon_0}$

freq dependent dielectric function

freq dependent atomic polarizability

$\vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega \Rightarrow \vec{D}(t) \neq \epsilon \vec{E}(t)$, but rather

ϵ complex $\Rightarrow \vec{D}_\omega$ and \vec{E}_ω are not in general in phase with each other

$\vec{D}(t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(t') \tilde{\epsilon}(t-t')$
∧ F.T. of $\epsilon(\omega)$

$\vec{D}(t)$ and $\vec{E}(t)$ are non-locally related in time

⇒ Maxwell's equations look very complicated when expressed in terms of time. For example:

assume $\mu = \mu_0$, $\vec{J}_{free} = 0$, then Ampere's law is

$$\mu_0 \vec{\nabla} \times \vec{B} = \frac{\partial \vec{D}}{\partial t} = \int \frac{dt'}{2\pi} \vec{E}(t')$$

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B}(\vec{r}, t) = \frac{\partial \vec{D}(\vec{r}, t)}{\partial t} = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(\vec{r}, t') \frac{d}{dt} \tilde{\epsilon}(t-t')$$

becomes an integro-differential equation when expressed in terms of \vec{B} and \vec{E} .

(Alternatively, Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, would become an integro-differential equation if expressed in terms of \vec{H} and \vec{D} .)

Do the fields \vec{E} and \vec{B} inside such a material with an $\epsilon(\omega)$ obey the wave equation when $\rho_f = 0$, $\vec{j}_f = 0$?

For simplicity we take a material with no magnetic response

$$\mu = \mu_0 \quad \rightarrow \quad \vec{H} = \frac{\vec{B}}{\mu_0}$$

Ampere's law when $\vec{j}_f = 0$ is

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \rightarrow \quad \vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$

use $\vec{D}(\vec{r}, t) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \vec{E}(\vec{r}, t') \tilde{\epsilon}(t-t')$

where $\tilde{\epsilon}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \epsilon(\omega)$
is F.T. of $\epsilon(\omega)$

Then

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{\partial}{\partial t} \int \frac{dt'}{2\pi} \vec{E}(\vec{r}, t') \tilde{\epsilon}(t-t') = \mu_0 \int \frac{dt'}{2\pi} \vec{E}(\vec{r}, t') \frac{\partial \tilde{\epsilon}(t-t')}{\partial t}$$

Take curl of both sides

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{B} = \mu_0 \int \frac{dt'}{2\pi} \left[\vec{\nabla} \times \vec{E}(\vec{r}, t') \right] \frac{\partial \tilde{\epsilon}(t-t')}{\partial t}$$

use $\vec{\nabla} \times \vec{E}(\vec{r}, t') = -\frac{\partial \vec{B}(\vec{r}, t')}{\partial t'}$ Faraday's law

$$-\nabla^2 \vec{B}(\vec{r}, t) = -\mu_0 \int \frac{dt'}{2\pi} \frac{\partial \vec{B}(\vec{r}, t')}{\partial t'} \frac{\partial \tilde{\epsilon}(t-t')}{\partial t}$$

use $\frac{\partial \tilde{\epsilon}(t-t')}{\partial t} = -\frac{\partial \tilde{\epsilon}(t-t')}{\partial t'}$

$$-\nabla^2 \vec{B}(\vec{r}, t) = \mu_0 \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \frac{\partial \vec{B}(\vec{r}, t')}{\partial t'} \frac{\partial \tilde{\epsilon}(t-t')}{\partial t'}$$

integrate by parts

$$-\nabla^2 \vec{B}(\vec{r}, t) = -\mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 \vec{B}(\vec{r}, t')}{\partial t'^2} \tilde{\epsilon}(t-t')$$

$$\nabla^2 \vec{B}(\vec{r}, t) - \mu_0 \int \frac{dt'}{2\pi} \frac{\partial^2 \vec{B}(\vec{r}, t')}{\partial t'^2} \tilde{\epsilon}(t-t') = 0$$

looks somewhat like the wave equation, except for the integration.

Suppose $\tilde{\epsilon}(t-t') = 2\pi\epsilon \delta(t-t')$ with ϵ a constant

Then we would get

$$\nabla^2 \vec{B}(\vec{r}, t) - \mu_0 \epsilon \int_{-\infty}^{\infty} dt' \frac{\partial^2 \vec{B}(\vec{r}, t')}{\partial t'^2} \delta(t-t') = 0$$

$$\Rightarrow \nabla^2 \vec{B}(\vec{r}, t) - \mu_0 \epsilon \frac{\partial^2 \vec{B}(\vec{r}, t)}{\partial t^2} = 0 \quad \text{The wave equation!}$$

So only if the response function $\tilde{\epsilon}(t) = 2\pi\epsilon \delta(t)$,
 i.e. it is instantaneous in time, we \vec{B} solve the wave equation
 with wave velocity $v = \frac{1}{\sqrt{\mu_0 \epsilon}}$

What $\epsilon(\omega)$ corresponds to $\tilde{\epsilon}(t-t') = 2\pi\epsilon \delta(t-t')$?

$$\epsilon(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \tilde{\epsilon}(t) \quad \text{definition of Fourier transform}$$

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega t} 2\pi\epsilon \delta(t) = \epsilon$$

So only when $\epsilon(\omega) = \epsilon$ is a constant, indep of ω ,
 will the fields obey the wave equation!

Moment

Calculation of $\tilde{x}(t)$ from $x(\omega)$ using contour integration

$x(t)$ is the response to $E(t) = \delta(t)$

For our simple model

$$\tilde{x}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{\omega^2 + i\gamma\omega - \omega_0^2}$$

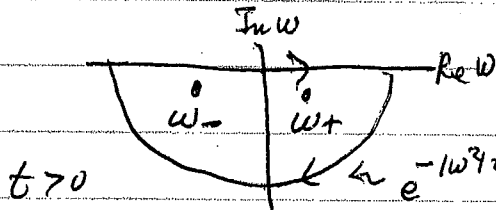
do by contour integration

$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+)(\omega - \omega_-)}$$

$$\omega_{\pm} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

poles at ω_{\pm} are in lower half complex plane.

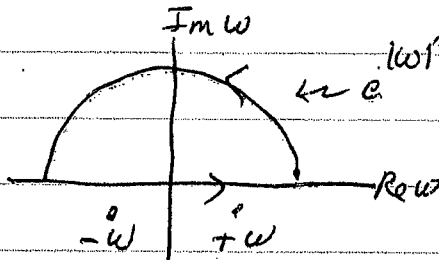
for $t > 0$, close contour in lower half plane



$t > 0$

contour encloses poles
+ get contribution
vanishes for $t > 0$

for $t < 0$, close contour in upper half plane



contour encloses
no poles \Rightarrow integral
vanishes

$$\tilde{x}(t) = 0 \quad \text{for } t < 0$$

causal response! No polarization until electric field turns on

For $t > 0$

$$\tilde{\alpha}(t) = \int d\omega e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega - \omega_+)(\omega - \omega_-)}$$

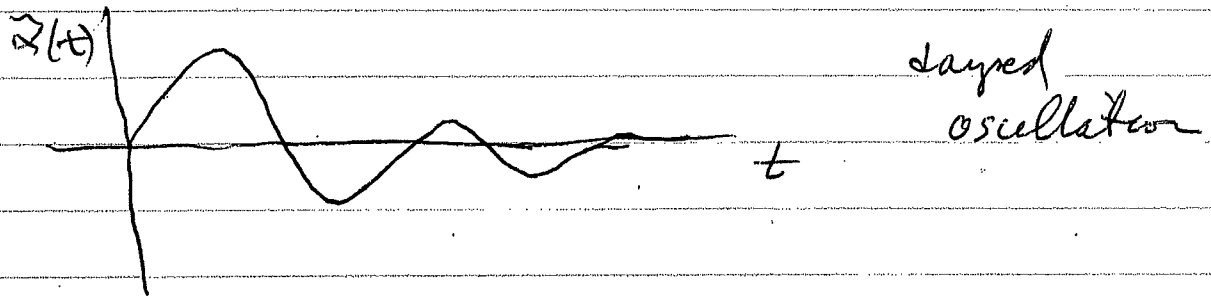
(-) sign since
go clockwise
around contour
from residue
theorem

$$= (-2\pi i) \frac{e^2}{m} (-1) \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

2 terms from the 2 poles

$$= 2\pi i \frac{e^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{\alpha}(t) = \begin{cases} 2\pi \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



Polarization density $\vec{P}_\omega = \epsilon_0 \chi_e(\omega) \vec{E}_\omega$ for harmonic oscillation

$$\chi_e(\omega) \approx \frac{N \alpha(\omega)}{\epsilon_0}$$

↑ atom density N

can use Clausius - Mossotti correction for denser materials

$$\Rightarrow \vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega \quad \epsilon(\omega) = \epsilon_0 (1 + \chi_e(\omega))$$

↑ freq dependent