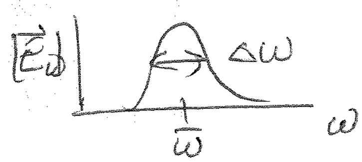
Our result $\vec{E}^2(r,t) = \vec{E}^2(0, t - \frac{dk}{d\omega} z)$ looks like we still preserve shape of wave – but this is due to the simplicity of our approximation. If we kept to next order, i.e. used $k(\omega) = k(\bar{\omega}) + \frac{dk}{d\omega}(\omega - \bar{\omega})$

$$+ \frac{1}{2}\frac{d^2k}{d\omega^2}(\omega-\bar{\omega})^2$$

one would find that the wave pulse changes shape as it propagates – in particular, it spreads.

A simple way to estimate this effect:

If pulse initially has width $\Delta\omega$ about $\bar{\omega}$, i.e. $\vec{E}_\omega$ looks like



there is a spread in group velocities

$$\Delta v_g \simeq \left|\frac{dv_g}{d\omega}\right| \Delta\omega = \left|\frac{d}{d\omega}\left(\frac{1}{dk/d\omega}\right)\right| \Delta\omega$$

$$= \frac{1}{(dk/d\omega)^2}\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega = v_g^2 \left|\frac{d^2k}{d\omega^2}\right|\Delta\omega$$

So if pulse take a time $T = z/\bar{v}_g$ to reach point z from the origin, there is also a spread in arrival times

$$\Delta T = \Delta(z/v_g) = \frac{z}{v_g^2}\Delta v_g = z\left|\frac{d^2k}{d\omega^2}\right|\Delta\omega$$

ΔT gives a spreading of width of the wave pulse, that grows linearly with the distance z traveled.

For a pulse of width $\Delta\omega$, the width in time is

$$\Delta t \sim \frac{1}{\Delta\omega} \quad \text{(like uncertainty principle in QM)}$$

$$\Rightarrow \Delta T \simeq z \left| \frac{d^2k}{d\omega^2} \right| \frac{1}{\Delta t}$$

$\Rightarrow$ the sharper the pulse is initially, (ie the smaller Δt) the faster it spreads as it travels (ie the larger ΔT is).

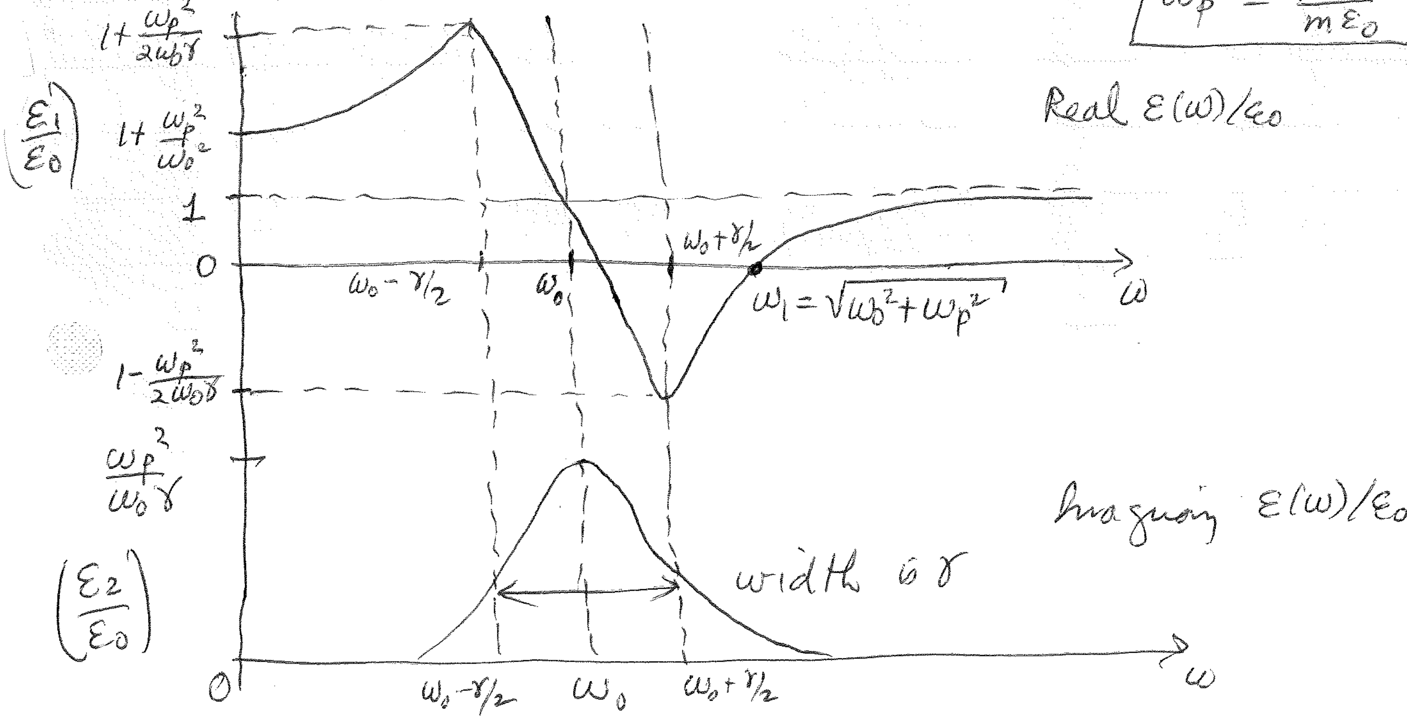
For our simple model

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\Rightarrow \quad \frac{\varepsilon_1}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$ real part ε

$$\frac{\varepsilon_2}{\varepsilon_0} = \frac{Ne^2}{m\varepsilon_0} \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}$$ Imaginary part ε

$$\boxed{\omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0}}$$ plasma freq



as $\left(\frac{\gamma}{\omega_0}\right) \to 0$, width of resonence decreases
height of peaks diverges

Notes for sketch $\varepsilon_1/\varepsilon_0$

max and min of ~~[illegible]~~ occur when ~~[illegible]~~ $\frac{\partial(\varepsilon_1/\varepsilon_0)}{\partial\omega} = 0$

$$\Rightarrow \left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right](-2\omega) - (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)(-2\omega)+2\omega\gamma^2\right]=0$$

$$(\omega_0^2-\omega^2)^2+\omega^2\gamma^2 - 2(\omega_0^2-\omega^2)^2 + (\omega_0^2-\omega^2)\gamma^2 = 0$$

$$(\omega_0^2-\omega^2)^2 = \omega_0^2\gamma^2$$

$$|\omega_0^2-\omega^2| = \omega_0\gamma$$

$$|\omega_0-\omega|(\omega_0+\omega) = \omega_0\gamma$$

for sharp resonance, peaks are when $\frac{\omega-\omega_0}{\omega_0} \ll 1 \rightarrow \omega_0+\omega \approx 2\omega_0$

$$\Rightarrow |\omega_0-\omega|\,2\omega_0 = \omega_0\gamma$$

$$|\omega_0-\omega| = \frac{\gamma}{2} \Rightarrow \boxed{\omega-\omega_0 = \pm\frac{\gamma}{2}}$$ location of max and min

$\Rightarrow$ width of resonance $= \gamma$

zero's of ε_1 define $\omega_p^2 \equiv \frac{Ne^2}{m\varepsilon_0}$

$$0 = 1 + \omega_p^2 \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$$\Rightarrow (\omega^2-\omega_0^2)^2 - \omega_p^2(\omega^2-\omega_0^2) + \omega^2\gamma^2 = 0$$

For the zero near the resonance, $\omega^2\gamma^2 \rightarrow \omega_0^2\gamma^2$ is good approx

$\omega^2-\omega_0^2 \rightarrow (\Delta\omega)2\omega_0$, $\Delta\omega \equiv \omega-\omega_0$

$$(\Delta\omega)^2 4\omega_0^2 - \Delta\omega\, 2\omega_0\omega_p^2 + \omega_0^2\gamma^2 = 0$$

$$(\Delta\omega)^2 - \frac{\omega_p^2}{2\omega_0}\Delta\omega + \frac{\gamma^2}{4} = 0$$

for $\omega_p \gg \omega_0$, $\Delta\omega \simeq \frac{\gamma^2\omega_0}{2\omega_p^2} = \frac{\gamma}{2}\underbrace{\left(\frac{\gamma}{\omega_0}\right)\left(\frac{\omega_0}{\omega_p}\right)^2}_{\text{both small}}$

generally true

shift of resonance small compared to width of resonance

For the zero above the resonance at ω_1

$$(\omega_1^2-\omega_0^2)^2 - \omega_p^2(\omega_1^2-\omega_0^2) + \omega_1^2\gamma^2 = 0$$

↑ small so ignore

$$\Rightarrow \quad \omega_1^2-\omega_0^2 = \omega_p^2$$

$$\omega_1^2 = \omega_0^2+\omega_p^2 \simeq \omega_p^2 \quad \text{when } \omega_p \gg \omega_0$$

<u>max of ~~...~~ ε_2</u>

$$\varepsilon_2 = \omega_p^2 \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

peak when $\dfrac{\partial \varepsilon_2}{\partial\omega} = 0 \Rightarrow ((\omega_0^2-\omega^2)^2+\omega^2\gamma^2)\gamma - \omega\gamma\left[2(\omega_0^2-\omega^2)(-2\omega) + 2\omega\gamma^2\right] = 0$

$$\Rightarrow (\omega_0^2-\omega^2)^2\gamma + 4\omega^2\gamma(\omega_0^2-\omega^2) - \omega^2\gamma^3 = 0$$

near resonance,

$$(\omega_0^2-\omega^2) = \Delta\omega(2\omega_0) = \frac{\omega^2\gamma^3}{4\omega^2\gamma} = \frac{\gamma^2}{4}$$

$$\Delta\omega = \frac{\gamma^2}{8\omega_0} \quad \text{small} \Rightarrow \text{peak at} \approx \omega_0$$

$$\frac{\varepsilon_2}{\varepsilon_0}(\omega_0) = \frac{\omega_p^2}{\omega\gamma}$$

half height at ω such that $\dfrac{\varepsilon_2}{\varepsilon_0}(\omega) = \dfrac{\omega_p^2}{2\omega\gamma}$

$$\Rightarrow \frac{1}{2\omega\gamma} = \frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2} \Rightarrow (\omega_0^2-\omega^2)^2+\omega^2\gamma^2 = 2\omega^2\gamma^2$$

$$\omega_0^2-\omega^2 = \pm\omega\gamma$$

for <u>sharp</u> resonance $\Delta\omega(2\omega_0) = \pm\omega_0\gamma$

$$\Delta\omega \simeq \pm\frac{\gamma}{2}$$

width of resonance peak in $\dfrac{\varepsilon_2}{\varepsilon_0}$ is γ.

$$k = k_1 + i k_2 = \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_1}{\varepsilon_0} + i \frac{\varepsilon_2}{\varepsilon_0}}$$

want to express k_1 and k_2 in terms of ε_1 and ε_2

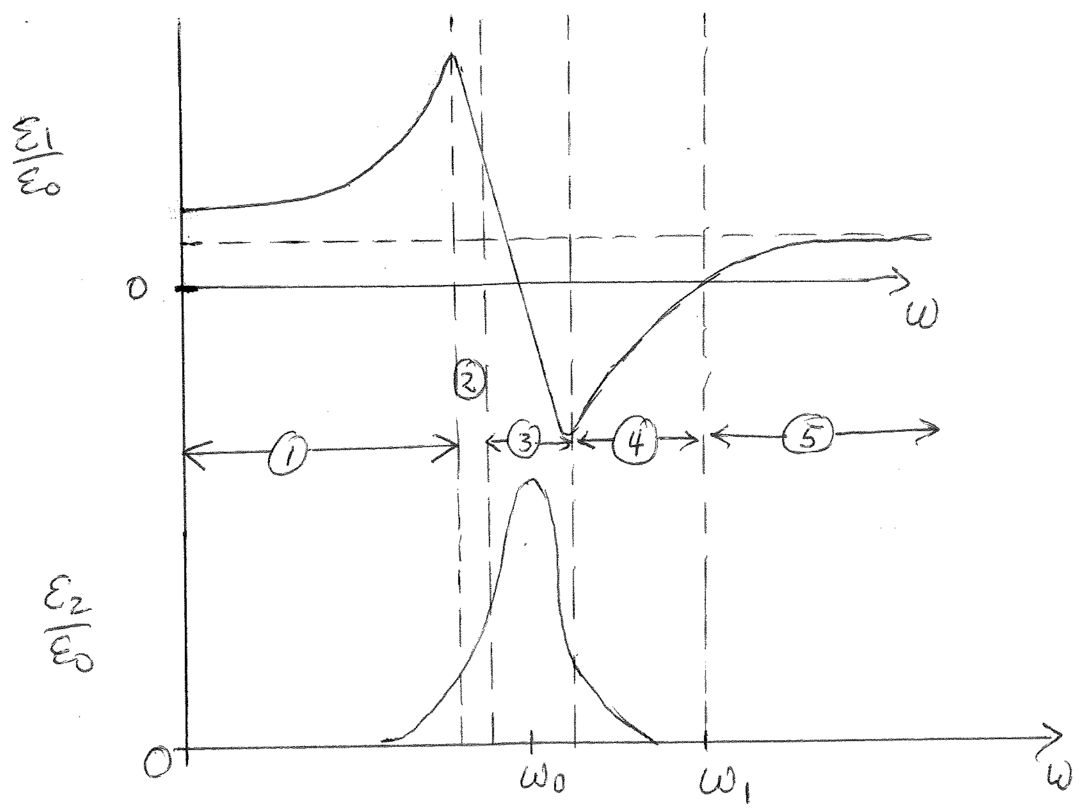
$$k^2 = k_1^2 - k_2^2 + 2 i k_1 k_2 = \frac{\omega^2}{c^2} \frac{\varepsilon_1}{\varepsilon_0} + i \frac{\omega^2}{c^2} \frac{\varepsilon_2}{\varepsilon_0}$$

equate real and imaginary pieces, and solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right) \right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c} \left[\frac{1}{2} \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_0}\right)^2 + \left(\frac{\varepsilon_2}{\varepsilon_0}\right)^2} - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right) \right]^{1/2}$$

Regions of different behavior



Regions (1) and (5) : transparent propagation

$\varepsilon_1 > 0 \qquad \varepsilon_1 \gg \varepsilon_2$

$$k_1 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\sqrt{1+\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

use $\sqrt{1+x} \simeq 1+\frac{x}{2}$ small x

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{\varepsilon_1}{\varepsilon_0} + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} \simeq \pm \frac{\omega}{c}\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$$

$$k_2 = \pm \frac{\omega}{c}\left[\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_0}\right)\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1\varepsilon_0}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right)$$

so $k_2 \ll k_1$ small attenuation $\Rightarrow$ transparent propagation

index of refraction $n = \frac{ck_1}{\omega} \cong \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$

$\frac{dn}{d\omega} > 0 \Rightarrow$ normal dispersion

phase velocity $v_p = \frac{\omega}{k_1} = \frac{c}{n} = c\sqrt{\frac{\varepsilon_0}{\varepsilon_1}}$

in region (1) $\frac{\varepsilon_1}{\varepsilon_0} > 1 \Rightarrow v_p < c$

in region (5) $\frac{\varepsilon_1}{\varepsilon_0} < 1 \Rightarrow v_p > c$! (but $v_g < c$ always)

Region ②: Similar to region ①, except that

$$\frac{dn}{d\omega} < 0 \Rightarrow \underline{\text{anomalous dispersion}}$$

Region ③: $\omega \approx \omega_0$ resonant absorption

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$

for a sharp resonance $\frac{\gamma}{\omega_0} \ll 1$ (typically $\omega_p \gg \omega_0$)

So: $\varepsilon_2 \gg \varepsilon_1$

$$k_1 = \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \sqrt{1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} \left(1 + \frac{1}{2} \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \right) + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}}$$

$$k_2 \simeq \pm \frac{\omega}{c} \left[\frac{1}{2} \frac{\varepsilon_2}{\varepsilon_0} + \frac{1}{4} \frac{\varepsilon_1^2}{\varepsilon_2 \varepsilon_0} - \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{\varepsilon_2}{2\varepsilon_0}}$$

$k_1 \simeq k_2 \Rightarrow$ strong attenuation

wave is exciting atoms near their resonant frequency ω_0 $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave. Wave decays rapidly ~~within~~ (factor e^{-1}) within one wavelength of propagation.

Region ④: $\varepsilon_1 < 0$, $|\varepsilon_1| \gg \varepsilon_2$ total reflection

* width of this region is $\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$
increases with atomic density (as $\omega_p \gg \omega_0$)

$$k_1 = \pm \frac{\omega}{c} \Big[\underbrace{\frac{1}{2} \left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2^2}{|\varepsilon_1| \varepsilon_0} + \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0}}_{\text{cancel}} \Big]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}} \, \frac{\varepsilon_2}{2|\varepsilon_1|}$$

$$k_2 = \pm \frac{\omega}{c} \left[\frac{1}{2} \left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2^2}{|\varepsilon_1| \varepsilon_0} - \frac{1}{2} \frac{\varepsilon_1}{\varepsilon_0} \right]^{1/2} \qquad \frac{\varepsilon_1}{\varepsilon_0} = - \left| \frac{\varepsilon_1}{\varepsilon_0} \right|$$

$$= \pm \frac{\omega}{c} \left[\left| \frac{\varepsilon_1}{\varepsilon_0} \right| + \frac{1}{4} \frac{\varepsilon_2}{|\varepsilon_1| \varepsilon_0} \right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c} \sqrt{\frac{|\varepsilon_1|}{\varepsilon_0}}$$

So $\dfrac{k_2}{k_1} = \dfrac{2|\varepsilon_1|}{\varepsilon_2} \gg 1$

wave vector k is almost pure imaginary ~~(since $\frac{\varepsilon_2}{\varepsilon_0} \ll 1$)~~
Wave decays exponentially $\to 0$ before traveling
even one wavelength into material.

We will see that this is a region of <u>total reflection</u>
Since $\omega \gg \omega_0$, not at resonance, material is
not absorbing much energy from wave. The
strong attenuation is due to the <u>destructive
interference</u> between the wave and the induced
fields of the polarized atoms.

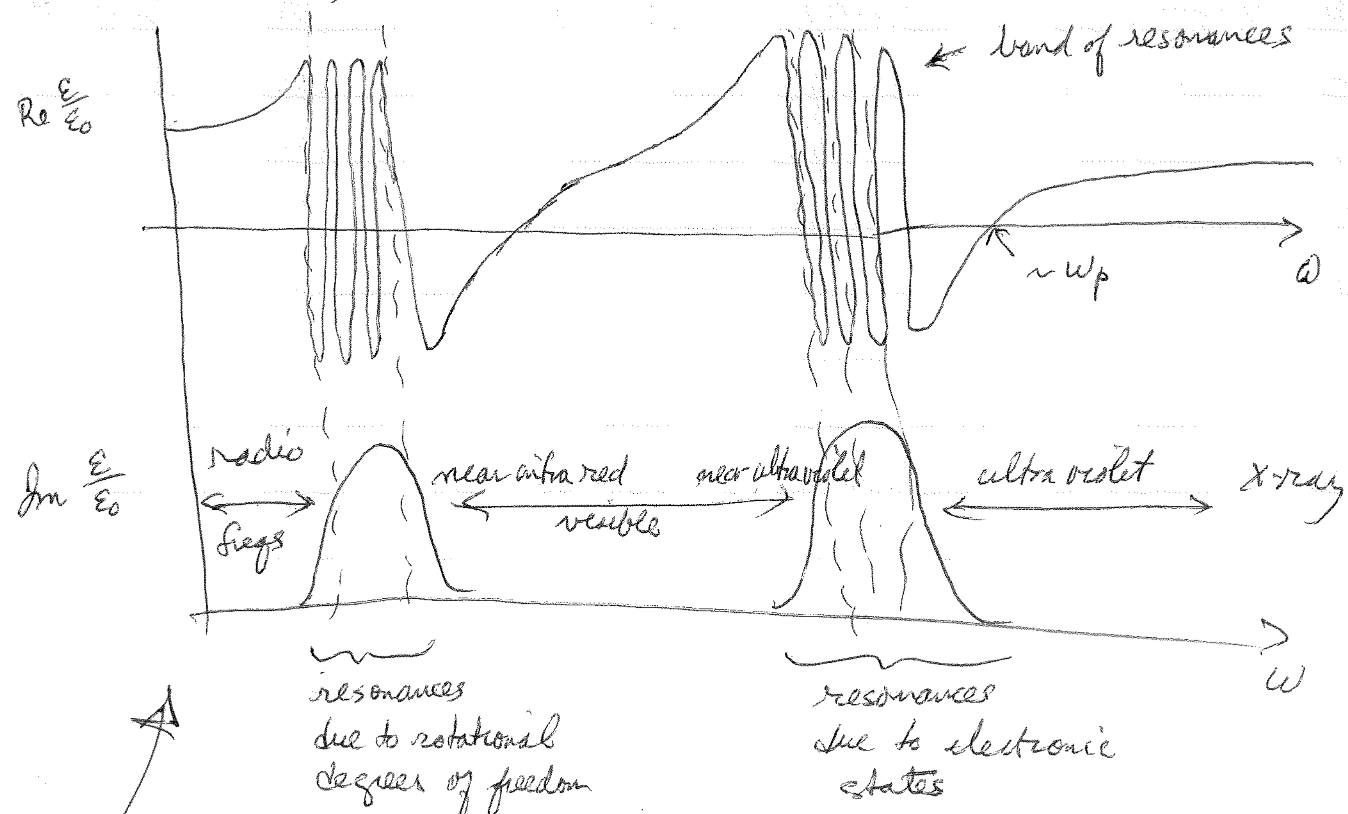
Our single model was

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \leftarrow \text{single resonance at } \omega \approx \omega_0$$

A more realistic model of an atom or molecule would give many resonances

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are the energy spacings between quantized electron ~~et~~ energy levels with an allowed electric dipole transition.



for a typical molecular gas

$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

$$\omega_p = c\sqrt{\frac{N_A}{\varepsilon_0}\frac{e^2}{mc^2}}\sqrt{\frac{N}{N_A}}$$

$$\frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8\times10^{-13}\ \text{cm}$$

$c = 3\times10^{10}$ cm/sec

$N_A = 6\times10^{23}$ cm^{-3} Avogadro's #

$$\omega_p = 4.4\times10^{16}\sqrt{\frac{N}{N_A}}\ \text{sec}^{-1}$$

$$\hbar\omega_p = 185\sqrt{\frac{N}{N_A}}\ \text{ev}$$

typical densities for H_2O or other $\overset{\text{dielectric}}{\text{liquid}}$ $\frac{N}{N_A} \simeq .05$

$\hbar\omega_p \simeq 40$ ev

compared to $\hbar\omega_0 \sim$ ev

for a metal, typical densities $\frac{N}{N_A} \simeq \frac{5\times10^{22}\ \text{cm}^{-3}}{6\times10^{23}\ \text{cm}^{-3}} \simeq 0.1$

$\omega_p \simeq 10^{16}$ sec^{-1}

$\hbar\omega_p \simeq 58$ ev