② Behavior at high freq

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \left(\frac{\omega_p}{\omega}\right)^2 \qquad \omega_p^2 = \frac{Ne^2}{m\varepsilon_0}$$ plasma freq

$\varepsilon(\omega)$ is real $\quad (\varepsilon_2 \ll \varepsilon_1)$

1) If $\omega > \omega_p$, then $\varepsilon > 0$

$\Rightarrow$ transparent propagation $\qquad k_1 = \frac{\omega}{c}\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

k is pure real $\qquad k_2 \approx 0$

2) If $\omega < \omega_p$, then $\varepsilon < 0$

$\Rightarrow$ total reflection $\qquad k_1 = 0$

$$k_2 = \frac{\omega}{c}\sqrt{\frac{\mu}{\mu_0}\left(\frac{\omega_p^2}{\omega^2} - 1\right)}$$

k is pure imaginary

plasma freq ω_p gives cross over between reflection + transparent propagation.

$\tau \sim 10^{-14}$ sec for typical metal

$\omega_p \simeq 10^{16}\ \text{sec}^{-1}$ for most metals

$$\lambda_p \equiv \frac{2\pi c}{\omega_p} \sim 3\times 10^3\ \text{Å}$$ (visible is $\lambda \sim 5\times10^3$ Å)

Example: The ionosphere is a layer of charged gas surrounding the earth. In may respects the charged gas behaves like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio propagates through ionosphere + escapes into space

Explains why you can pick up AM stations from far away - they are reflected back by ionosphere - but you only pick up local FM stations - they do not get reflected by ionosphere.

What about longitudinal modes? (ie H_ω, E_ω <u>not</u> $\perp \vec{k}$)

<u>magnetic field</u>

$i\mu \vec{k}\cdot\vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k}$ $\underline{\underline{\text{or}}}$ $\vec{k}=0$ uniform magnetic field

Faraday $\quad i\vec{k}\times\vec{E}_\omega = i\omega\mu\vec{H}_\omega \Rightarrow \omega = 0$

$\quad\quad\quad \overset{\parallel}{0}$ as $\vec{k}=0$

($\vec{H}\perp\vec{k}$ would be transverse mode so longitudinal mode must have $k=0$ and so $\omega=0$.

So only possible longitudinal magnetic field is a spatially uniform, constant in time $\vec{H}$.

<u>electric field</u>

$$i\,\varepsilon(\omega)\vec{k}\cdot\vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k}, \ \underline{\underline{\text{or}}}\ \vec{k}=0, \ \underline{\underline{\text{or}}}\ \varepsilon(\omega)=0\ !$$

we can satisfy all Maxwell's equations for a $\vec{E}_\omega \parallel \vec{k}$, provided $\varepsilon(\omega)=0$, and by above, $\vec{H}_\omega = 0$ for this mode.

$i\vec{k}\times\vec{E}_\omega = i\omega\mu\vec{H}_\omega$ — both sides vanish.
LHS $=0$ as $\vec{E}_\omega \parallel \vec{k} \Rightarrow \vec{k}\times\vec{E}_\omega = 0$
RHS $=0$ as $\vec{H}_\omega = 0$

$i\vec{k}\times\vec{H}_\omega = -i\omega\varepsilon(\omega)\vec{E}_\omega$ — LHS $=0$ as $\vec{H}_\omega=0$
RHS $=0$ as $\varepsilon(\omega)=0$

$i\mu\vec{k}\cdot\vec{H}_\omega = 0$ — satisfied as $\vec{H}_\omega = 0$

So we can have a longitudinal ~~oscillation~~ $\vec{E}$ provided $\varepsilon(\omega)=0$

frequencies of longitudinal mode given by $\varepsilon(\omega) = 0$.

<u>low freq</u> $\omega \ll \omega_0$, $\omega\tau \ll 1$ N_a = density of polarizable atoms

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_b}{\varepsilon_0} + \frac{i\sigma}{\varepsilon_0 \omega} \simeq 1 + \frac{N_a e^2}{m\varepsilon_0} + \frac{i\sigma_0}{\varepsilon_0 \omega} = \frac{1}{\varepsilon_0}\left(\varepsilon_b(0) + \frac{i\sigma_0}{\omega}\right)$$

$\frac{\varepsilon}{\varepsilon_0} = 0$ when $\omega = \frac{-i\sigma_0}{\varepsilon_b(0)}$

$$\Rightarrow \vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_\omega e^{-\sigma_0 t/\varepsilon_b(0)} e^{i\vec{k}\cdot\vec{r}}$$

$\Rightarrow$ if set up a longitudinal $\vec{E}$ field, it decays to zero exponentially fast, with decay time $\frac{\varepsilon_b(0)}{\sigma_0}$

Consistant with our assumption that $\vec{E} = 0$ inside a conductor for electrostatics.

(electrostatic fields are <u>always</u> longitudinal

$\vec{E} = -\vec{\nabla} V \Rightarrow$ ~~$\vec{E}_k = i\vec{k} V_k$ for Fourier component~~

$\vec{E} \sim -i\vec{k} V_k e^{i\vec{k}\cdot\vec{r}}$ $\quad \vec{E} \sim \vec{k}$

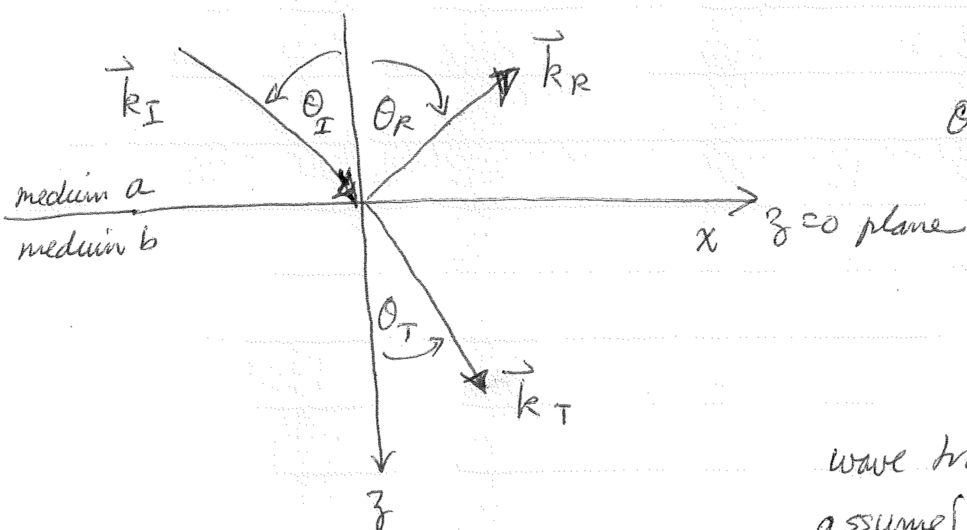
<u>high freq</u> $\omega \gg 1/\tau$, $\omega \gg \omega_0$

then $\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$ $\quad \omega_p^2 = \frac{Ne^2}{\varepsilon_0 m}$

$\varepsilon(\omega) = 0$ when $\omega = \omega_p$ the plasma freq

longitudinal oscillation of $\vec{E}$ (and ρ) at $\omega = \omega_p$

Reflection and Transmission (Refraction) of waves



θ_I = angle of incidence
θ_R = angle of reflection
θ_T = angle of transmission (refraction)

wave traveling from a to b.

assume $\begin{cases} \mu_a \text{ and } \mu_b \text{ are real} \\ \varepsilon_a \text{ real} \\ \varepsilon_b \text{ may be complex} \end{cases}$

$$\vec{E}_I = \vec{E}_{\omega I}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)}$$

$$\vec{E}_R = \vec{E}_{\omega R}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$$

$$\vec{E}_T = \vec{E}_{\omega T}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$$

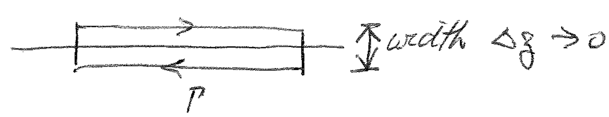
similarly for $\vec{H}_I$, $\vec{H}_R$, $\vec{H}_T$

in each media $k^2 = \frac{\omega^2}{c^2}\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0} = \omega^2 \mu \varepsilon$

$$k_I^2 = \omega_I^2 \mu_a \varepsilon_a, \quad k_R^2 = \omega_R^2 \mu_a \varepsilon_a, \quad k_T^2 = \omega_T^2 \mu_b \varepsilon_b$$

boundary conditions at interface

Faraday $\quad \vec{\nabla} \times \vec{E}_\omega - i\omega\mu \vec{H}_\omega = 0$



$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{E}_\omega) = \int_S d\vec{a} \cdot \vec{H}_\omega\, i\omega\mu \to 0 \quad \text{as } \Delta z \to 0$$

surface bounded by Γ

$$\oint_\Gamma d\vec{\ell} \cdot \vec{E}_\omega \Rightarrow (\vec{E}_{above} - \vec{E}_{below}) \cdot d\vec{\ell} = 0$$

$\Rightarrow$ tangential component of $\vec{E}$ is continuous across interface

Ampere $\quad \nabla \times \vec{H}_\omega = -i\omega\varepsilon \vec{E}_\omega \quad$ (assuming no free current at boundary)

same argument as for $\vec{E}$ $\Rightarrow$ tangential component of $\vec{H}$ is continuous at interface

apply to $\vec{E}$ at interface: For $\hat{t}$ any unit vector in xy plane

$$\hat{t} \cdot (\vec{E}_I + \vec{E}_R) = \hat{t} \cdot \vec{E}_T$$

$\Rightarrow$ for any $\vec{\rho}$ in xy plane at $z=0$, and any time t

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{i(\vec{k}_I \cdot \vec{\rho} - \omega_I t)} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{i(\vec{k}_R \cdot \vec{\rho} - \omega_R t)}$$

$$= \hat{t} \cdot \vec{E}_{\omega T}\, e^{i(\vec{k}_T \cdot \vec{\rho} - \omega_T t)}$$

true for any $\vec{\rho}$, so consider at $\vec{\rho} = 0$

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{-i\omega_I t} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{-i\omega_R t} = \hat{t} \cdot \vec{E}_{\omega T}\, e^{-i\omega_T t}$$

must be true for all t $\Rightarrow$ $\boxed{\omega_I = \omega_R = \omega_T}$

all freq's equal

Now consider for $\rho \neq 0$, at $t=0$.

$$\hat{t} \cdot \vec{E}_{\omega I}\, e^{i\vec{k}_I \cdot \vec{\rho}} + \hat{t} \cdot \vec{E}_{\omega R}\, e^{i\vec{k}_R \cdot \vec{\rho}} = \hat{t} \cdot \vec{E}_{\omega T}\, e^{i\vec{k}_T \cdot \vec{\rho}}$$