must be true for all $\vec{\rho}$ $\Rightarrow$ $\vec{k}_I \cdot \vec{\rho} = \vec{k}_R \cdot \vec{\rho} = \vec{k}_T \cdot \vec{\rho}$ all ρ

$\Rightarrow$ projections of $\vec{k}_I$, $\vec{k}_R$, $\vec{k}_T$ in xy plane are all equal. only z-components of $\vec{k}_I$, $\vec{k}_R$, $\vec{k}_T$ may differ

Choose coordinates as in diagram so that all $\vec{k}$'s lie in xz plane.

$$k_{Ix} = k_{Rx} \Rightarrow |\vec{k}_I| \sin\theta_I = |\vec{k}_R| \sin\theta_R$$

$$|\vec{k}_I| = \omega\sqrt{\mu_a \varepsilon_a} = |\vec{k}_R| \Rightarrow \boxed{\theta_I = \theta_R}$$

<u>angle of incidence = angle of reflection</u>

If $\sqrt{\varepsilon_b}$ is also real (ie in region of transparent propagation)

then $|\vec{k}_T| = \omega\sqrt{\mu_b \varepsilon_b}$

$$k_{Ix} = k_{Tx} \Rightarrow |\vec{k}_I| \sin\theta_I = |\vec{k}_T| \sin\theta_T$$

$$\omega\sqrt{\mu_a \varepsilon_a} \sin\theta_I = \omega\sqrt{\mu_b \varepsilon_b} \sin\theta_T$$

$$\frac{\sin\theta_T}{\sin\theta_I} = \sqrt{\frac{\mu_a \varepsilon_a}{\mu_b \varepsilon_b}}$$

in terms of index of refraction $n \equiv \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\varepsilon}\, c}{\omega}$

$n \equiv \frac{c}{v_p}$ $\qquad$ $n = \sqrt{\mu\varepsilon}\, c = \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$

$$\boxed{\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_a}{n_b}}$$

Snell's law — true for all types of waves, not just EM waves

$$\sin\theta_T = \frac{n_a}{n_b}\sin\theta_I$$

If $n_a > n_b$, then $\theta_T > \theta_I$

in this case,

when θ_I is too large, we will have $\frac{n_a}{n_b}\sin\theta_I > 1$

and there is no solution for θ_T

$\Rightarrow \vec{E}_T = 0$, there is no transmitted wave.

this is called "total internal reflection" — wave does not exit medium a.

critical angle $\theta_c = \arcsin\left(\frac{n_b}{n_a}\right) \leftarrow$ [the bigger n_a/n_b, the smaller θ_c

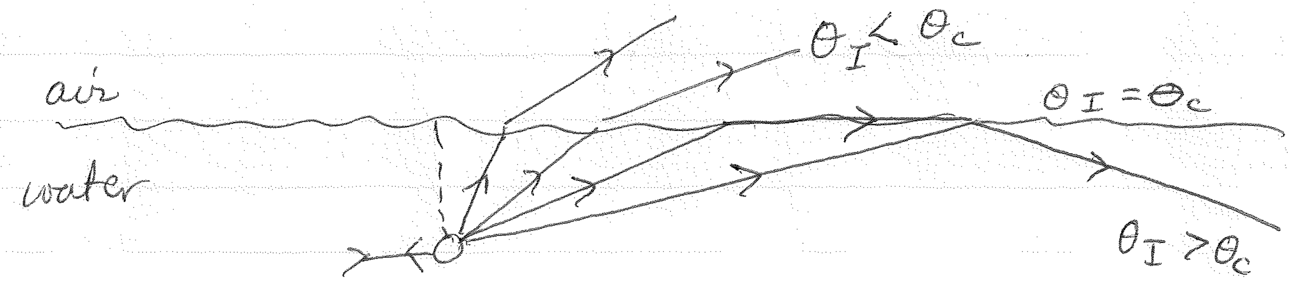
total internal reflection whenever $\theta_I > \theta_c$

total internal reflection usually happens as one goes from a denser to a less dense ~~mater~~ medium as

$$\left(\frac{n}{c}\right)^2 = \mu\epsilon \sim \mu\epsilon_0\left(1 + \frac{Ne^2}{m\epsilon_0}\right)$$ where N is density of polarizable atoms (m is electron mass).

total internal reflection is why diamonds sparkle! diamond has big $n \Rightarrow$ small $\theta_c \Rightarrow$ light bounces around inside diamond getting totally internally reflected many times, before it is able to escape.

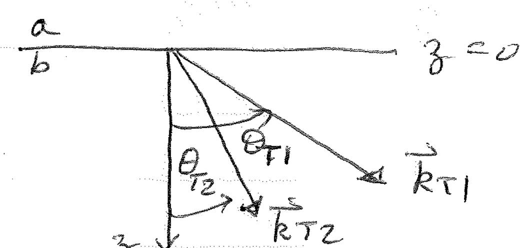
Can also experience total internal reflection in the swimming pool:



$n_{water} > n_{air}$ swimmer for $\theta_I > \theta_c$, can't see out of water

when $\theta_I = \theta_c$, transmitted wave travels parallel to interface

<u>More general case</u>: $\sqrt{\epsilon_b}$ can be complex $\Rightarrow \vec{k}_T$ is complex



$\vec{k}_T = \vec{k}_{T1} + i\vec{k}_{T2}$

$k_{T1} \equiv |\vec{k}_{T1}|$, $k_{T2} \equiv |\vec{k}_{T2}|$

$\vec{k}_{T1}$ and $\vec{k}_{T2}$ need not be in same direction!

$$\vec{k}_{Tx} = \vec{k}_{Ix} \Rightarrow k_{T1} \sin\theta_{T1} + i k_{T2} \sin\theta_{T2} = k_I \sin\theta_I$$

equate real and imaginary pieces $\Rightarrow$ $\boxed{\begin{matrix} k_{T1}\sin\theta_{T1} = k_I \sin\theta_I \\ k_{T2}\sin\theta_{T2} = 0 \end{matrix}}$

$\Rightarrow \boxed{\theta_{T2} = 0}$

ie attenuation factor for the transmitted wave is of the form $e^{-k_{T2} z}$

$\Rightarrow$ planes of constant <u>amplitude</u> are <u>parallel</u> to the interface, no matter what the angle of incidence θ_I.

planes of constant <u>phase</u> are $\perp$ to $\vec{k}_{T1}$

Now we solve for k_{T1} and k_{T2} and θ_{T1}

Dispersion relation in medium 2: $k_T^2 = \omega^2 \mu_b \varepsilon_b$

$$k_T^2 = (\vec{k}_{T1} + i\vec{k}_{T2})^2 = k_{T1}^2 - k_{T2}^2 + 2i\vec{k}_{T1}\cdot\vec{k}_{T2}$$
$$= k_{T1}^2 - k_{T2}^2 + 2i k_{T1} k_{T2} \cos\theta_{T1} \qquad (\text{since } \theta_{T2} = 0)$$
$$= \omega^2 \mu_b (\varepsilon_{b1} + i\varepsilon_{b2})$$

equate real and imaginary parts of both sides

$$k_{T1}^2 - k_{T2}^2 = \omega^2 \mu_b \varepsilon_{b1}$$
$$2 k_{T1} k_{T2} = \frac{\omega^2 \mu_b \varepsilon_{b2}}{\cos\theta_{T1}}$$

} same equations as when we considered propagation in an infinite dielectric, only then $\theta_{T1} = 0$

Consider the above as two equations for two unknowns k_{T1} and k_{T2}. Solve for k_{T1} and k_{T2} in terms of $\cos\theta_{T1}$

$$k_{T1} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2}$$

$$k_{T2} = \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} - \frac{\varepsilon_{b1}}{2}\right]^2$$

If $\theta_{T1} = 0$, this is the same as our earlier result

Finally we use our boundary condition to determine θ_{T1}

$$k_{T1} \sin\theta_{T1} = k_I \sin\theta_I$$

$$k_I = \omega\sqrt{\mu_a \varepsilon_a} = \frac{\omega}{c}\sqrt{\frac{\mu_a \varepsilon_a}{\mu_0 \varepsilon_0}} = \frac{\omega}{c} n_a$$

↑ index of refraction

$$k_{T1} = \frac{k_I \sin\theta_I}{\sin\theta_{T1}} = \frac{\omega}{c} \frac{n_a \sin\theta_I}{\sin\theta_{T1}}$$

$$= \omega\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2}$$

$$\Rightarrow \boxed{n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{\varepsilon_{b1}}{2}\right]^{1/2} \sin\theta_{T1}}$$

↑ determines angle of transmission θ_{T1} in terms of angle of incidence θ_I and the physical parameters $n_a, \mu_b, \varepsilon_{b1}, \varepsilon_{b2}$ of the two materials

Cases ① If material b is transparent, ie $\varepsilon_{b2} \ll \varepsilon_{b1}$

define $n_b = \sqrt{\frac{\mu_b \varepsilon_{b1}}{\mu_0 \varepsilon_0}} = \sqrt{\mu_b \varepsilon_{b1}}\, c$

then $n_a \sin\theta_I = n_b \sin\theta_{T1}\left[\frac{1}{2\varepsilon_{b1}}\sqrt{\varepsilon_{b1}^2 + \frac{\varepsilon_{b2}^2}{\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$

$$= n_b \sin\theta_{T1}\left[\frac{1}{2}\sqrt{1 + \frac{\varepsilon_{b2}^2}{\varepsilon_{b1}^2\cos^2\theta_{T1}}} + \frac{1}{2}\right]^{1/2}$$

expand the $\sqrt{1+\delta} \approx 1 + \delta/2$

$$\simeq n_b \sin\theta_{T1}\left[\frac{1}{2} + \frac{\varepsilon_{b2}^2}{4\varepsilon_{b1}^2\cos^2\theta_{T1}} + \frac{1}{2}\right]^{1/2}$$

$$= n_b \sin\theta_{T1}\left[1 + \frac{\varepsilon_{b2}^2}{4\varepsilon_{b1}^2\cos^2\theta_{T1}}\right]^{1/2}$$

expand the $\sqrt{}$

$$n_a \sin\theta_I \simeq n_b \sin\theta_{T1}\left[1 + \underbrace{\frac{\varepsilon_{b2}^2}{8\varepsilon_{b1}^2\cos^2\theta_{T1}}}_{\text{small correction to Snell's law}}\right]$$

when $\frac{\varepsilon_{b2}}{\varepsilon_{b1}} \ll 1$, we can solve above equation "interatively" to get approximate result

$$n_a \sin\theta_I = n_b \sin\theta_{T1}\left[1 + \text{small}\right]$$

$$\Rightarrow \sin\theta_{T1} \approx \frac{n_a}{n_b}\sin\theta_I \Rightarrow \cos^2\theta_{T1} \approx 1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I$$

so to next order

$$n_b \sin\theta_{T1} \simeq \frac{n_a \sin\theta_I}{1 + \frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2\left[\frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}\right]}$$

$$\simeq n_a \sin\theta_I\left[1 - \underbrace{\frac{1}{8}\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2 \frac{1}{1 - \frac{n_a^2}{n_b^2}\sin^2\theta_I}}_{\text{this term is } > 0 \text{ so} \ldots}\right]$$

$$\lesssim n_a \sin\theta_I$$

Result is that θ_{T1} is <u>smaller</u> than one would predict from Snell's law.

The correction is of order $O\left(\frac{\varepsilon_{b2}}{\varepsilon_{b1}}\right)^2$.

Case (2) medium b is a good conductor or a region of resonant absorption of a dielectric so $\varepsilon_{b2} \gg \varepsilon_{b1}$

Now, to lowest order we will approx $\varepsilon_{b1} \approx 0$ then

$$n_a \sin\theta_I = c\sqrt{\mu_b}\left[\frac{1}{2}\frac{\varepsilon_{b2}}{\cos\theta_{T1}}\right]^{1/2} \sin\theta_{T1}$$

$$\boxed{n_a \sin\theta_I = c\sqrt{\frac{\mu_b \varepsilon_{b2}}{2}}\ \frac{\sin\theta_{T1}}{\sqrt{\cos\theta_{T1}}}}$$

determines θ_{T1} in terms of θ_I

In this case our result for θ_{T1} looks nothing like Snell's Law.

$\Rightarrow$ Snell's Law only holds if both media are transparent at the frequency of interest