Similarly, if we could find the "Green's function" for the $\Box^2$ operator, ie a function $G(\vec{r}-\vec{r}', t-t')$ that solved

$$\Box^2 G(\vec{r}-\vec{r}', t-t') = -4\pi\, \delta(\vec{r}-\vec{r}')\, \delta(t-t')$$

then the solutions to $\Box^2 V = -\rho/\varepsilon_0$, $\Box^2 \vec{A} = -\mu_0 \vec{J}$, would be

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t')\, \rho(\vec{r}', t')$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, G(\vec{r}-\vec{r}', t-t)\, \vec{J}(r', t')$$

How to find $G(\vec{r}-\vec{r}', t-t')$? Use Fourier transf method

$$G(\vec{r},t) = \int d^3k \int d\omega\, \tilde{G}(\vec{k},\omega)\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}$$

$$\delta(\vec{r}) = \int d^3k\, \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^3} \qquad \text{from HW}$$

$$\delta(t) = \int d\omega\, \frac{e^{-i\omega t}}{2\pi}$$

substitute into $\Box^2 G(\vec{r},t) = -4\pi\, \delta(\vec{r})\, \delta(t)$

$$\int d^3k \int d\omega\, \tilde{G}(k,\omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t} = -4\pi \int d^3k \int d\omega\, \frac{e^{i\vec{k}\cdot\vec{r}}\, e^{-i\omega t}}{(2\pi)^4}$$

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

$$\frac{\partial^2}{\partial t^2} e^{-i\omega t} = -\omega^2 e^{-i\omega t}$$

$$\int d^3k \int d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \int d^3k \int d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \frac{(-4\pi)}{(2\pi)^4}$$

equate Fourier coefficients

$$\Rightarrow \left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\tilde{G}(k,\omega) = \frac{-4\pi}{(2\pi)^4}\frac{c^2}{(\omega^2-c^2k^2)}$$

$$G(\vec{r},t) = \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\tilde{G}(k,\omega)$$

$$= \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d\omega\, \frac{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}{(2\pi)^4} \frac{(-)4\pi c^2}{(\omega+ck)(\omega-ck)}$$

integrand diverges when $\omega = \pm ck$

Can evaluate using methods of complex contour integration (see complex variables course)

$$G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{c}{r}\delta(r-ct) & t>0 \end{cases} \quad \text{where } r = |\vec{r}|$$

$$\boxed{G(\vec{r},t) = \begin{cases} 0 & t<0 \\ \frac{1}{r}\delta\left(t-\frac{r}{c}\right) & t>0 \end{cases}} \quad \text{as } \delta(ax) = \frac{\delta(x)}{a}$$

$G(\vec{r},t)$ has a reasonable form:

1) $G(\vec{r},t) \sim \delta(|\vec{r}|-ct)$ $\Rightarrow$ response travels with speed c
response from source at $\vec{r}'=0$, $t'=0$,
is only felt at ~~time t = r~~ position $\vec{r}$
at time $t = \frac{r}{c}$ later.

2) If take $c \to \infty$, $G(\vec{r},t) \to \frac{\delta(t)}{r}$ response instantaneous
and $\frac{1}{r}$ is Greens function of ∇^2
expected as $\lim_{c\to\infty} \Box^2 = \nabla^2$

Explicit check that $G = \frac{c}{r}\delta(r-ct)$ solves $\Box^2 G = -4\pi\,\delta(\vec{r})\,\delta(t)$

$$\nabla^2(ab) = a\nabla^2 b + b\nabla^2 a + 2\vec{\nabla}a\cdot\vec{\nabla}b$$

$$\nabla^2\left[\frac{c}{r}\delta(r-ct)\right] = \frac{c}{r}\nabla^2\delta(r-ct) + 2\,\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) + \delta(r-ct)\,\nabla^2\left(\frac{c}{r}\right)$$

use $\nabla^2\delta(r-ct) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\delta(r-ct)\right)$ in spherical coords

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\delta'(r-ct)\right) = \frac{1}{r^2}\left[2r\,\delta'(r-ct) + r^2\delta''(r-ct)\right]$$

$$= \frac{2}{r}\delta'(r-ct) + \delta''(r-ct)$$ here $\delta'(x) = \frac{d\delta(x)}{dx}$ etc.

$$\vec{\nabla}\left(\frac{c}{r}\right)\cdot\vec{\nabla}\delta(r-ct) = \left(\frac{-c}{r^2}\right)\left(\delta'(r-ct)\right)$$ (evaluated in spherical coords)

$$\nabla^2\left(\frac{c}{r}\right) = -4\pi\,\delta(\vec{r})\,c$$

so $\nabla^2\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{c}{r}\left(\frac{2}{r}\delta'(r-ct) + \delta''(r-ct)\right)$

$$- \frac{2c}{r^2}\,\delta'(r-ct) \quad - 4\pi c\,\delta(\vec{r})\,\delta(r-ct)$$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta\left(t-\frac{r}{c}\right)$$ using $\delta(r-ct) = \frac{\delta\left(t-\frac{r}{c}\right)}{c}$

$$= \frac{c}{r}\,\delta''(r-ct) \quad - 4\pi\,\delta(\vec{r})\,\delta(t)$$

since $r=0$ because of $\delta(r)$ term

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{c}{r}\,\delta(r-ct)\right] = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{c}{r}(-c)\,\delta'(r-ct)\right]$$

$$= -\frac{1}{r}\frac{\partial}{\partial t}\,\delta'(r-ct) \;=\; \frac{c}{r}\,\delta''(r-ct)$$

so $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{c}{r}\,\delta(r-ct)\right) = \frac{c}{r}\delta''(r-ct) - 4\pi\,\delta(\vec{r})\,\delta(t)$

$$-\frac{c}{r}\,\delta''(r-ct)$$

$$= -4\pi\,\delta(\vec{r})\,\delta(t)$$ as desired

$$V(\vec{r},t) = \int \frac{d^3r'}{4\pi\varepsilon_0}\;\frac{\rho(r',t')}{|\vec{r}-\vec{r}'|}$$

where $t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$ "retarded time" depends on $\vec{r}$ and $\vec{r}'$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\;\frac{\vec{J}\left(\vec{r}',\, t' = t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

general solution to inhomogeneous wave equation

Summary

The Green's function solves

$$\Box^2 G(\vec{r}-\vec{r}\,', t-t') = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

Therefore the solution to

$$\Box^2 V(\vec{r},t) = -\frac{\rho(\vec{r},t)}{\varepsilon_0}$$

is

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\ \rho(\vec{r}\,',t')\, G(\vec{r}-\vec{r}\,', t-t')$$

check:

$$\Box^2 V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\ \rho(\vec{r}\,',t')\, \Box^2 G(\vec{r}-\vec{r}\,', t-t')$$

since $\Box^2$ acts on $\vec{r}$ and t not on $\vec{r}\,'$ and t'

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\, \rho(\vec{r}\,',t')\,(-4\pi)\,\delta(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

integrate over the δ-functions

$$= -\frac{\rho(r,t)}{\varepsilon_0}$$

Similarly, the solution to $\Box^2 \vec{A}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t)$ is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{\infty} dt'\ \vec{J}(\vec{r}\,',t')\, G(\vec{r}-\vec{r}\,', t-t')$$

Now our solution for the Greens function was

$$G(\vec{r}-\vec{r}\,', t-t') = \begin{cases} 0 & t < t' \\ \dfrac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\!\left(t-t' - \dfrac{|\vec{r}-\vec{r}\,'|}{c}\right) & t > t' \end{cases}$$

So we can write our solution for $V(\vec{r},t)$ as

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \int_{-\infty}^{t} dt'\, \rho(\vec{r}',t')\, \frac{1}{|\vec{r}-\vec{r}'|}\, \delta\!\left(t - t' - \frac{|\vec{r}-\vec{r}'|}{c}\right)$$

do integral over the δ-function

$$= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \, \frac{\rho\left(\vec{r}',\, t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

or $$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} d^3r' \, \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

where here t' is the "retarded" time

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$ depends on $\vec{r}'$

Similarly

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} d^3r' \, \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$$

$$t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

Radiation by localized oscillating charge distribution

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\, dt'\, \vec{J}(\vec{r}',t') \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

For pure harmonic oscillating current $\vec{J}(\vec{r},t) = Re\{\vec{J}(r,\omega)\, e^{-i\omega t}\}$

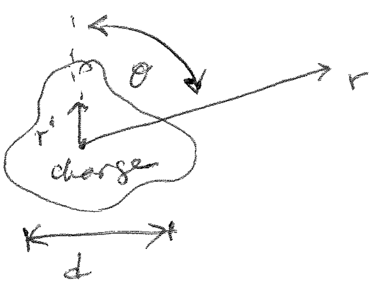
Resulting $\vec{A}$ will oscillate at same freq ω $\vec{A}(r,t) = Re\{\vec{A}(\vec{r},\omega)\, e^{-i\omega t}\}$

$$\vec{A}(\vec{r},\omega)\, e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3r'\, dt'\, \vec{J}(\vec{r}',\omega)\, e^{-i\omega t'} \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},\omega)\, e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3r'\, \vec{J}(\vec{r}',\omega)\, e^{-i\omega t} \frac{e^{+i\omega\frac{|\vec{r}-\vec{r}'|}{c}}}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\int d^3r'\, \vec{J}(\vec{r}',\omega) \frac{e^{i\omega|\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume $\vec{J}(\vec{r}',\omega) \simeq 0$ for $|\vec{r}'| > d$, ie charge is localized with region of size d about origin.



Approx ①

for $r >> d$, ie far from sources

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$$

$$= r\sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\theta}$$

$$\simeq r\left(1 - \frac{r'}{r}\cos\theta + O\left(\frac{r'}{r}\right)^2\right)$$

$$\simeq r - \vec{r}'\cdot\hat{r}$$

$\hat{r}$ is unit vector along $\vec{r}$

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{J}(\vec{r}',\omega)\, e^{ik(r - \vec{r}'\cdot\hat{r})}}{r - \vec{r}'\cdot\hat{r}} \qquad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3r' \frac{\vec{J}(\vec{r}',\omega)\, e^{-ik\vec{r}'\cdot\hat{r}}}{1 - \frac{\hat{r}\cdot\vec{r}'}{r}} \qquad \leftarrow \text{expand } \frac{1}{1-\delta} \sim 1+\delta$$

$$= \frac{\mu_0}{4\pi} \left(\frac{e^{ikr}}{r}\right) \int d^3r' \vec{J}(\vec{r}',\omega)\, e^{-ik\hat{r}\cdot\vec{r}'} \left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

when combine with factor $e^{-i\omega t}$ $\quad (\vec{A}(\vec{r},t) = \vec{A}(\vec{r},\omega)e^{-i\omega t})$
this piece gives spherical waves

$$\frac{e^{i(kr-\omega t)}}{r}$$

$\Rightarrow$ oscillating charge <u>radiates</u> outgoing spherical electromagnetic waves

$\int d^3r\, \vec{J} \cdots$ term will determine angular dependence of the radiation

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, \vec{J}(\vec{r}',\omega)\, e^{-ik\hat{r}\cdot\vec{r}'}\left(1+\frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

<u>approximation ②</u> long wavelength limit $\lambda \gg d$

or $kd \ll 1 \Rightarrow \frac{\omega}{c}d \ll 1$ or $\frac{d}{\tau} \ll c$

Since d is the maximum distance over which charge moves in period of oscillation τ, we see that $kd \ll 1$

$\Rightarrow v \ll c$ where $v \approx \frac{d}{\tau}$ is characteristic velocity with which the charges move

$\Rightarrow \lambda \gg d$ is a non-relativistic approx.

$kd \ll 1 \Rightarrow e^{-ik\hat{r}\cdot\vec{r}'} \simeq 1 - ik\hat{r}\cdot\vec{r}' +$ higher order

$$\vec{A}(\vec{r},\omega) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, \vec{J}(\vec{r}',\omega)\overbrace{\left[1 + \hat{r}\cdot\vec{r}'\left(\frac{1}{r} - ik\right)\right]}^{(1-ik\hat{r}\cdot\vec{r}')(1+\frac{\hat{r}\cdot\vec{r}'}{r})}$$

+ higher order terms in $\frac{d}{r}$ or kd.

$$\vec{A}(\vec{r},\omega) \doteq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{\vec{I}_1 + \left(\frac{1}{r} - ik\right)\vec{I}_2\right\}$$

where $\vec{I}_1 \equiv \int d^3r'\, \vec{J}(\vec{r}',\omega)$

$\vec{I}_2 \equiv \int d^3r'\, \hat{r}\cdot\vec{r}'\, \vec{J}(\vec{r}',\omega)$