~~Inverse transform obtained by taking $v \to -v$ in above~~

$$\cancel{\begin{cases} ct = \gamma ct' + \gamma\left(\frac{v}{c}\right) x' \\ x = \gamma\left(\frac{v}{c}\right) ct' + \gamma x' \end{cases}}$$

4 - vectors

4-position: $x_\mu = (x_1, x_2, x_3, ict)$ $\qquad x_4 \equiv ict$

summation convention $x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu^2 = r^2 - c^2t^2$ Lorentz invariant scalar

- sum over repeated indicies

— has same value in all interial frames

Lorentz transf for $K \to K'$ where K' moves with $v\hat{x}$ as seen by K.

$$\left.\begin{aligned} x_1' &= \gamma\left(x_1 + i\left(\tfrac{v}{c}\right)x_4\right) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma\left(x_4 - i\left(\tfrac{v}{c}\right)x_1\right) \end{aligned}\right\}$$

linear transf, can be represented by a matrix

or $x_\mu' = a_{\mu\nu}(L)\, x_\nu$

↑ matrix of Lorentz transformation L

$$a(L) = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

inverse: $x_\mu = a_{\mu\nu}(L^{-1})\, x_\nu'$

$a_{\mu\nu}(L^{-1})$ is given by taking $v \to -v$ in $a_{\mu\nu}(L)$

we see $a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)$

inverse = transpose $\Rightarrow$ "orthogonal"

More generally

Since x_μ^2 is Lorentz invariant scalar,

$$x_\mu'^2 = a_{\mu\nu}(L)\, a_{\mu\lambda}(L)\, x_\nu x_\lambda = x_\lambda^2$$

$$\Rightarrow \quad a_{\mu\nu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\nu\mu}(L)\, a_{\mu\lambda}(L) = \delta_{\nu\lambda}$$

$$\Rightarrow \quad a^t_{\mu\nu} = a^{-1}_{\mu\nu}(L) \quad \text{transpose = inverse}$$

$a_{\mu\nu}$ is 4x4 orthogonal matrix

a matrix whose transpose equals its inverse is called an ortogonal matrix.

If L_1 is a Lorentz transf from K to K'
L_2 is a Lorentz transf from K' to K''
Then the Lorentz transf from K to K'' is given by the matrix

$$a(L_2 L_1) = a(L_2)\, a(L_1)$$

if $L_1 \equiv L$ and $L_2 = L^{-1}$ so $L_2 L_1 = I$ identity

$$\Rightarrow \quad a^{-1}(L) = a(L^{-1})$$

particle on trajectory $\vec{r}(t)$

$dx_1 = x_1(t+dt) - x_1(t)$
etc

4-differential

$$dx_\mu = (dx_1, dx_2, dx_3, icdt)$$

$$-(dx_\mu)^2 \equiv c^2 ds^2 = c^2 dt^2 - dr^2 \quad \text{Lorentz invariant scalar}$$

$$ds^2 = dt^2\left[1 - \frac{1}{c^2}\left(\frac{dx_1}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_2}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{dx_3}{dt}\right)^2\right]$$

$$ds^2 = \frac{dt^2}{\gamma^2} \qquad \vec{v} = \frac{d\vec{r}}{dt}$$

$$\boxed{ds = \frac{dt}{\gamma}}$$

proper time interval
ds is the same in all inertial frames.

A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does x_μ

4-velocity $\quad u_\mu \equiv \dfrac{dx_\mu}{ds} \equiv \overset{\circ}{x}_\mu$ dot indicates derivative with respect to s

$$= \gamma \frac{dx_\mu}{dt}$$

since dx_μ is a 4-vector and ds is Lorentz invarient scalar, then $\dfrac{dx_\mu}{ds}$ is a 4-vector,

space components $\vec{u} = \gamma \vec{v}$

$u_4 = ic\gamma \qquad u_\mu = \gamma(\vec{v}, ic)$

$$u_\mu u_\mu = \gamma^2 v^2 - c^2\gamma^2 = \gamma^2(v^2 - c^2)$$
$$= \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2$$
Lorentz invarient scalar

4-acceleration $\quad a_\mu \equiv \dfrac{du_\mu}{ds} = \gamma \dfrac{du_\mu}{dt}$

4-gradient $\quad \dfrac{\partial}{\partial x_\mu} \equiv \left(\vec{\nabla}, \dfrac{-i}{c}\dfrac{\partial}{\partial t}\right) = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}, \dfrac{\partial}{\partial x_4}\right)$

where $x_4 = ict$

proof $\dfrac{\partial}{\partial x_\mu}$ is a 4-vector

by chain rule: $\dfrac{\partial}{\partial x_\mu'} = \dfrac{\partial x_\lambda}{\partial x_\mu'}\dfrac{\partial}{\partial x_\lambda} \longrightarrow$ but $\dfrac{\partial x_\lambda}{\partial x_\mu'} = a_{\lambda\mu}(L^{-1})$

$= a_{\mu\lambda}(L)$

inverse = transpose

So $\dfrac{\partial}{\partial x_\mu'} = a_{\mu\lambda}(L)\dfrac{\partial}{\partial x_\lambda}$

so transforms same as x_μ

$$\left(\frac{\partial}{\partial x_\mu}\right)^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$
wave equation operator!

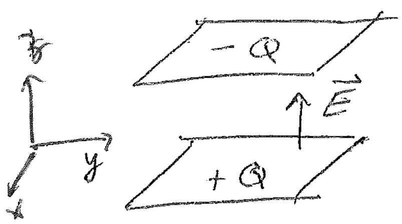
inner products

If u_μ and v_μ are 4-vectors, then $u_\mu v_\mu$ is Lorentz invarient scalar

Maxwell's Equations in Relativistic Form

How do $\vec{E}$ and $\vec{B}$ transform under Lorentz transformation?
$\vec{E}$ and $\vec{B}$ have much more complicated transformation laws than position 4-vector $X_\mu = (\vec{r}, ict)$

Example: parallel plate capacitor at rest in K
plates have area A, charge Q



$$\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z} \quad \text{uniform} \qquad \frac{Q}{A} = \sigma \text{ surface charge den}$$

$$\vec{B} = 0$$

In K', moving with $\vec{v} = v\hat{y}$ wrt K, y dimension of plates is contracted by factor $1/\gamma$ (FitzGerald Contraction)

$$\sigma' = \frac{Q}{A'} = \frac{\gamma Q}{A} = \gamma\sigma$$

$$\vec{E}' = \frac{Q}{A'\varepsilon_0}\hat{z} = \frac{\gamma Q}{A\varepsilon_0}\hat{z} = \gamma\vec{E} \qquad \vec{E} \text{ is along } \hat{z} \perp \vec{v}.$$

This is different than transf law for $\vec{r}$.
Under L.T. components of $\vec{r} \perp \vec{v}$ do not change
But components of $\vec{E} \perp \vec{v}$ do change

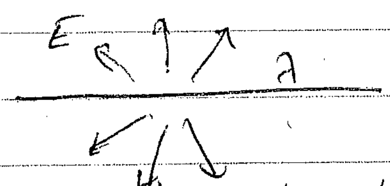
Also, moving surface charge σ' gives rise to surface current density $\Rightarrow$ there will be magnetic field $\vec{B}'$ in frame K'. $\Rightarrow$ Lorentz transf must couple together the components of $\vec{E}$ and $\vec{B}$.

Electromagnetism

Clearly $\vec{E}$ + $\vec{B}$ must transform into each other under Lorentz transf.

in inertial frame K

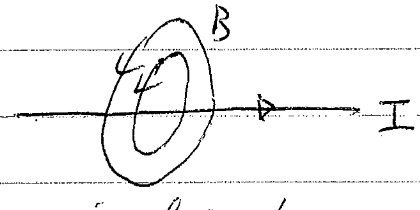
stationary line charge λ



cylindrical outward electric field

no B-field

in frame K' moving with $\vec{v}$ ∥ to wire



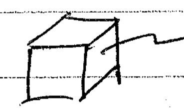
moving line charge gives current
⇒ B circulating around wire
as well as outward radial E

Lorentz force

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

What is the velocity $\vec{v}$ here? velocity with respect to what inertial frame? Clearly $\vec{E}$ and $\vec{B}$ must change from one inertial frame to another if this force law can make sense.

Charge density, current density



ΔV contains charge ΔQ

Consider charge ΔQ contained in a vol ΔV.
ΔQ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame

$\Delta Q = \overset{\circ}{\rho}\, \Delta \overset{\circ}{V}$ $\overset{\circ}{\rho}$ is charge density in rest frame of charge
$\Delta \overset{\circ}{V}$ is volume of box in rest frame

$\overset{\circ}{\rho}$ is a Lorentz invarient scalar by definition
Now transform to another frame moving with velocity $\vec{v}$ with respect to the rest frame.

ΔQ remains the same.

$\Delta V = \dfrac{\Delta \overset{\circ}{V}}{\gamma}$ volume contracts in direction $\parallel$ to $\vec{v}$

$$\Rightarrow \rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta \overset{\circ}{V}} \gamma = \overset{\circ}{\rho} \gamma$$

spatial components of 4-velocity

current density is $\vec{j} = \rho \vec{v} = (\rho/\gamma)(\gamma \vec{v}) = \overset{\circ}{\rho}\, \vec{u}$

Define 4-current $j_\mu \equiv \overset{\circ}{\rho}\, u_\mu = \overset{\circ}{\rho}(\vec{u}, ic\gamma)$

spatial components of j_μ are $\vec{j} = \overset{\circ}{\rho}\vec{u} = \rho\vec{v}$ current density

temporal component of j_μ is $j_4 = ic\overset{\circ}{\rho}\gamma = ic\rho$ charge density

So $\boxed{j_\mu = (\vec{j}, ic\rho)}$

j_μ is a 4-vector since u_μ is a 4-vector and $\overset{\circ}{\rho}$ is Lorentz invarient scalar

length of the 4-current is $j_\mu j_\mu = |\vec{j}|^2 - c^2\rho^2 = \overset{\circ}{\rho}^2 u_\mu u_\mu$
$= -c^2 \overset{\circ}{\rho}^2$

Charge conservation

$$0 = \vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = \vec{\nabla}\cdot\vec{j} + \frac{\partial(ic\rho)}{\partial(ict)} = \vec{\nabla}\cdot\vec{j} + \frac{\partial j_4}{\partial x_4}$$

$\Rightarrow \boxed{\dfrac{\partial j_\mu}{\partial x_\mu} = 0}$ charge conservation in Lorentz covariant form

Equations for potentials in Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \Box^2 \vec{A} = -\mu_0 \vec{j} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = \Box^2 V = -\rho/\varepsilon_0 = -c^2\mu_0\rho$$

$$= -\mu_0 (ic\rho)\left(\frac{c}{i}\right)$$

So

$$= -\mu_0 j_4 \left(\frac{c}{i}\right)$$

$$\Box^2 \vec{A} = -\mu_0 \vec{j}$$

$$\Box^2 (iV/c) = -\mu_0 j_4$$

Define 4-potential $\boxed{A_\mu = (\vec{A}, iV/c)}$

$$\Rightarrow \quad \Box^2 A_\mu = -\mu_0 j_\mu \quad \text{equation for potentials}$$

$\Box^2 = \dfrac{\partial^2}{\partial x_\nu^2}$ is Lorentz invariant operator

So we can write the above as $\boxed{\dfrac{\partial^2 A_\mu}{\partial x_\nu^2} = -\mu_0 j_\mu}$

Lorentz gauge condition is

$$0 = \vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$$

$$= \vec{\nabla}\cdot\vec{A} + \frac{\partial (iV/c)}{\partial (ict)} = \vec{\nabla}\cdot\vec{A} + \frac{\partial A_4}{\partial x_4}$$

$$= \frac{\partial A_\mu}{\partial x_\mu}$$

So Lorentz Gauge condition is $\boxed{\dfrac{\partial A_\mu}{\partial x_\mu} = 0}$

Electric and Magnetic Fields

$$\vec{B} = \vec{\nabla} \times \vec{A} \implies \boxed{B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}}$$

where i, j, k are a cyclic permutation of 1,2,3

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$V = \frac{cA_4}{i}, \quad x_4 = ict$$

$$\implies E_i = -\frac{\partial\left(\frac{c}{i}A_4\right)}{\partial x_i} - \frac{\partial A_i}{\partial\left(\frac{x_4}{ic}\right)} = -\frac{c}{i}\frac{\partial A_4}{\partial x_i} - ic\frac{\partial A_i}{\partial x_4}$$

$$\boxed{\frac{E_i}{c} = i\left(\frac{\partial A_4}{\partial x_i} - \frac{\partial A_i}{\partial x_4}\right)}$$

has a similar form to B_i

We define the field strength tensor

$$\boxed{F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}} = -F_{\nu\mu}$$

4×4 antisymmetric 2^{nd} rank tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{pmatrix}$$

"curl" of a 4-vector is a 4×4 antisymmetric 2^{nd} rank tensor

4×4 antisymmetric 2^{nd} rank tensor has only 6 independent components – just the right number to specify the $\vec{E}$ and $\vec{B}$ fields!

$F_{\mu\nu}$ transforms under a Lorentz transformation just like a tensor (i.e. not like a vector)

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu}$$

use $A'_\nu = a_{\nu\lambda} A_\lambda$

$\frac{\partial}{\partial x'_\mu} = a_{\mu\sigma} \frac{\partial}{\partial x_\sigma}$

since A_μ and $\frac{\partial}{\partial x_\sigma}$ are both 4-vectors

$$F'_{\mu\nu} = a_{\nu\lambda} a_{\mu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma} a_{\nu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda}$$

$$= a_{\mu\sigma} a_{\nu\lambda} \left(\frac{\partial A_\lambda}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\lambda} \right)$$

$$\boxed{F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda}}$$

← transformation law for a 2nd rank tensor

In terms of matrix multiplication, and writing for the transpose of a matrix $a_{\nu\lambda} = a^t_{\lambda\nu}$, the above can b written as

$$F'_{\mu\nu} = a_{\mu\sigma} F_{\sigma\lambda} a^t_{\lambda\nu}$$

The above has the form of the product of three matrices

If we write out the above transformation law component by component we get the following transformation law for the $\vec{E}$ and $\vec{B}$ fields.
For a transformation from K to K', where K' moves with velocity $v\hat{x}$ as seen from K,

$$E'_1 = E_1 \qquad B'_1 = B_1$$

$$E'_2 = \gamma(E_2 - vB_3) \qquad B'_2 = \gamma\left(B_2 + \frac{v}{c^2} E_3\right)$$

$$E'_3 = \gamma(E_3 + vB_2) \qquad B'_3 = \gamma\left(B_3 - \frac{v}{c^2} E_2\right)$$

where $(1,2,3) = (x,y,z)$