The transformation law for an n^{th} rank tensor is

$$T'_{\mu_1,\mu_2,\ldots\mu_n} = a_{\mu_1\nu_1} a_{\mu_2 a_2} \cdots a_{\mu_n\nu_n} T_{\nu_1,\nu_2,\ldots,\nu_n}$$

Inhomogeneous Maxwell's Equations

Using the field strength tensor $F_{\mu\nu}$ we can write the inhomogeneous Maxwell's equations (ie the ones involving the sources ρ and $\vec{J}$) as follows:

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu}$$

$F_{\mu\nu}$ is a 4-tensor 2nd rank

$\frac{\partial}{\partial x_\nu}$ is a 4-vector

$\Rightarrow \frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector

Proof that $\frac{\partial F_{\mu\nu}}{\partial x_\nu}$ is a 4-vector. Using the transformation laws of $F_{\mu\nu}$ and $\frac{\partial}{\partial x_\nu}$ we get

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} a_{\nu\sigma} a_{\nu\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau}$$

write $\sum_\nu a_{\nu\sigma} a_{\nu\tau} = \sum_\nu a^t_{\sigma\nu} a_{\nu\tau}$

but since a is orthogonal $a^t = a^{-1}$ and $\sum_\nu a^t_{\sigma\nu} a_{\nu\tau} = \delta_{\sigma\tau}$

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\lambda} \delta_{\sigma\tau} \frac{\partial F_{\lambda\sigma}}{\partial x_\tau} = a_{\mu\lambda} \frac{F_{\lambda\sigma}}{\partial x_\sigma}$$

so transforms like a 4-vector

back to $\boxed{\dfrac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu}$

To see this is so, substitute in definition of $F_{\mu\nu}$ in terms of 4-potential A_μ

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2 A_\mu}{\partial x_\nu^2}$$

1st term = 0 by Lorentz gauge condition. So

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = -\frac{\partial^2 A_\mu}{\partial x_\nu^2} = -\Box^2 A_\mu = \mu_0 j_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu \Rightarrow \begin{cases} \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} & \text{spatial components} \\ \vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \rho/\varepsilon_0 & \text{temporal component} \end{cases}$$

We still need to have a Lorentz covariant way to write the homogeneous Maxwell Equations.

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Homogeneous Maxwell Equations

Construct the 3rd rank co-variant tensor

$$\boxed{\tilde{G}_{\mu\nu\lambda} \equiv \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}}$$

transforms as $\tilde{G}'_{\mu\nu\lambda} = a_{\mu\alpha} a_{\nu\beta} a_{\lambda\gamma} \tilde{G}_{\alpha\beta\gamma}$

$\tilde{G}_{\mu\nu\lambda}$ has in principle $4^3 = 64$ components

But can show that $\tilde{G}$ is antisymmetric in exchange of any two indicies

$$\tilde{G}_{\nu\mu\lambda} = \frac{\partial F_{\nu\mu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\lambda}}{\partial x_\nu} \qquad \text{but since } F_{\mu\nu} = -F_{\nu\mu}$$

$$= -\frac{\partial F_{\mu\nu}}{\partial x_\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x_\mu} - \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = -G_{\mu\nu\lambda}$$

$\Rightarrow$ $\tilde{G}_{\nu\mu\lambda} = 0$ if any two indicies are equal

$\Rightarrow$ there are only $\underline{\underline{4}}$ independent components of $G_{\mu\nu\lambda}$

these are

$$\tilde{G}_{123},\ \tilde{G}_{124},\ \tilde{G}_{134},\ \tilde{G}_{234}$$

all other components are just equal to $\pm$ one of these according to permutation of indicies.

The 4 homogeneous Maxwell equations can be written as

$$\boxed{\tilde{G}_{\mu\nu\lambda} = 0}$$

To see that above is true, substitute in for $F_{\mu\nu}$ in terms of potential A_μ in definition of $\tilde{G}$

$$\tilde{G}_{\mu\nu\lambda} = \underbrace{\frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} \underbrace{- \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda}}_{\text{cancell}} \underbrace{- \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu}}_{\text{cancell}} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda}}_{\text{cancell}}$$

also, one has

$$\tilde{G}_{123} = \frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_2} + \frac{\partial F_{23}}{\partial x_1} = \frac{\partial B_3}{\partial x_3} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_1}{\partial x_1} = 0$$

$$\tilde{G}_{123} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\tilde{G}_{412} = \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{12}}{\partial x_4} = \frac{i\,\partial E_1}{c\,\partial x_2} + \frac{-i\,\partial E_2}{c\,\partial x_1} + \frac{\partial B_3}{ic\,\partial t}$$

$$= \frac{i}{c}\left[\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial t}\right] = -\frac{i}{c}\left[(\vec{\nabla}\times\vec{E})_3 + \frac{\partial B_3}{\partial t}\right] = 0$$

this is the z-component of Faraday's law $\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$

$\tilde{G}_{413} = 0$ and $\tilde{G}_{423} = 0$ give x and y components of Faraday's law.

An alternative way to write the homogeneous Maxwell's Equations

Note: we can get the homogeneous Maxwell's equations from the inhomogeneous equations by making the substitutions

$$\vec{j}\to 0,\ \rho\to 0,\ \frac{\vec{E}}{c}\to\vec{B},\ \vec{B}\to -\frac{\vec{E}}{c}$$

So we define the <u>dual field strength tensor</u>

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_3/c & E_2/c & -iB_1 \\ E_3/c & 0 & -E_1/c & -iB_2 \\ -E_2/c & E_1/c & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix}$$

or equivalently if

generalization of the Levi-Civita symbol

$$\varepsilon_{\mu\nu\sigma\lambda} = \begin{cases} +1 & \text{if } \mu\nu\sigma\lambda \text{ is an even permutation of } 1234 \\ -1 & \text{if } \mu\nu\sigma\lambda \text{ is an odd permutation of } 1234 \\ 0 & \text{otherwise, ie any two indices equal} \end{cases}$$

then $\boxed{G_{\mu\nu} = \frac{1}{2i} \varepsilon_{\mu\nu\lambda\sigma} F_{\lambda\sigma}}$ psuedo-tensor gives wrong sign under parity transform.

Then $\boxed{\frac{\partial G_{\mu\nu}}{\partial x_\nu} = 0}$

gives the homogeneous Maxwell's equations

From $F_{\mu\nu}$ ad $G_{\mu\nu}$ we can construct the following Lorentz invariant scalars

$$\frac{1}{2} F_{\mu\nu} F_{\mu\nu} = B^2 - \frac{E^2}{c^2}$$

$$-\frac{1}{4} F_{\mu\nu} G_{\mu\nu} = \frac{\vec{B} \cdot \vec{E}}{c}$$

these have the same value in any inertial frame of reference!

$\Rightarrow$ 1) If $\vec{E} \perp \vec{B}$ ad $|\vec{B}| = \frac{|\vec{E}|}{c}$ in one frame of reference, then it is so in all frames of reference.
($\vec{E} \cdot \vec{B} = 0$ ad $|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2} = 0$)
This property is satisfied by EM waves in the vacuum

2) If in one frame $\vec{E} \cdot \vec{B} = 0$ and $\frac{E^2}{c^2} > B^2$, then there exists a frame in which $\vec{B}' = 0$. If in one frame $\vec{E} \cdot \vec{B} = 0$ and $B^2 > E^2/c^2$, then there exists a frame in which $\vec{E}' = 0$.

Relativistic Kinematics

4-momentum of a particle $p_\mu = m\,\mathring{x}_\mu = m u_\mu = (m\gamma\vec{v}, \, i m c \gamma)$

m is mass of particle as measured in the frame in which the particle is instantaneous at rest. m = "rest mass"

p_μ is a 4-vector since m is a scalar and u_μ is a 4-vector

$$p_\mu^2 = m^2 u_\mu^2 = -m^2c^2 \qquad \text{since } u_\mu^2 = -c^2$$

4-force $K_\mu = (\vec{K}, \, i K_0)$ also called "Minkowski force"

We guess that the relativistic generalization of Newton's 2nd law of motion is

$$m\frac{d^2 x_\mu}{ds^2} = K_\mu \quad \text{or} \quad m\frac{du_\mu}{ds} = K_\mu$$

$$\text{or} \quad \frac{dp_\mu}{ds} = K_\mu \qquad (p_\mu = m u_\mu = m\,\mathring{x}_\mu)$$

Now since $p_\mu^2 = -m^2c^2$ is a constant, we have

$$0 = \frac{d}{ds}(p_\mu^2) = 2p_\mu \frac{dp_\mu}{ds} = 2p_\mu K_\mu$$

$$\Rightarrow \quad p_\mu K_\mu = 0$$

$$p_\mu K_\mu = m\gamma\vec{v}\cdot\vec{K} - mc\gamma K_0 = 0$$

$$\text{so} \quad \boxed{K_0 = \frac{\vec{v}}{c}\cdot\vec{K}}$$

time component of 4-force is determined by the spacial components $\vec{K}$

Define the usual 3-force by

$$\frac{d\vec{p}}{dt} = \vec{F}$$ (we identify the Newtonian momentum $\vec{p}$ with the spatial components of p_μ)

$$\frac{d\vec{p}}{ds} = \vec{K}$$ spatial part of relativistic Newton's law

$$\frac{d\vec{p}}{ds} = \gamma \frac{d\vec{p}}{dt} = \gamma \vec{F}$$ since $ds = dt/\gamma$

$$\Rightarrow \boxed{\vec{K} = \gamma \vec{F}}$$ relation between spatial part of 4-force and the usual 3-force $\vec{F}$

$$\Rightarrow K_0 = \frac{\vec{v}}{c} \cdot \vec{K} = \gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

Consider now the 4-th component of Newton's equations

$$\frac{dp_4}{ds} = m\frac{du_4}{ds} = m\frac{d}{ds}(ic\gamma) = iK_0 = i\gamma \frac{\vec{v}}{c} \cdot \vec{F}$$

$$\Rightarrow \frac{d}{ds}(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}$$

$$d(m\gamma c^2) = \gamma \vec{v} \cdot \vec{F}\, ds = \gamma \vec{v} \cdot \vec{F} \frac{dt}{\gamma}$$

$$= \vec{v} \cdot \vec{F}\, dt = d\vec{r} \cdot \vec{F}$$

$\Rightarrow$

Work-energy theorem: $d(m\gamma c^2) = d\vec{r} \cdot \vec{F}$

↑ work done on particle

$\Rightarrow$ change in kinetic energy of particle

relativistic kinetic energy $\boxed{E = m\gamma c^2}$

$$p_4 = im\gamma c = iE/c$$

$$p_\mu = (\vec{p}, \frac{iE}{c})$$ momentum-energy 4-vector

$$\vec{p} = m\gamma\vec{v}$$

$$E = m\gamma c^2$$

For particles moving at non-relativistic speeds $v \ll c$

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \simeq \frac{mc^2}{1-\frac{v^2}{2c^2}} \simeq mc^2\left(1+\frac{v^2}{2c^2}\right)$$

$$\simeq mc^2 + \frac{1}{2}mv^2$$

non-relativistic kinetic energy

rest mass energy

$\frac{dp_\mu}{ds} = K_\mu$ is therefore both the relativistic analogue of Newton's 2nd law, but also the law of conservation of energy (ie the work-energy theorem)

Conservation of momentum and energy

① why is relativistic momentum $\vec{p} = m\gamma\vec{v}$ and not just $m\vec{v}$ as in non-relativistic case?

Because we want momentum to be conserved in all frames of reference, $\vec{p}$ must be the spatial part of a 4-vector. We see this as follows.

Suppose momentum was $m\vec{v}$. For a collection of particles, conservation of momentum would mean

$$(*) \qquad \sum_i m_i \vec{v}_i(t_1) = \sum_i m_i \vec{v}_i(t_2)$$

for any times t_1 and t_2

If $(*)$ holds in one frame of reference K, and we now transform to another frame of reference K' moving with velocity $\vec{w}$ wrt K, we would find that in K', $(*)$ is no longer satisfied

ie $\sum_i m_i \vec{v}_i'(t_1) \neq \sum_i m_i \vec{v}_i'(t_2)$ — $\vec{v}_i'$ related to $\vec{v}_i$ and $\vec{w}$ via relativistic law for addition of velocities

see Griffiths ~~chapt 10.2~~ example 12.6

However, for the 4-momentum, if

$$P_\mu^{tot}(t_1) = \sum_i P_{\mu i}(t_1) = \sum_i P_{\mu i}(t_2) = P_\mu^{tot}(t_2)$$

in frame K, then $P_\mu'^{tot}(t_1) = P_\mu'^{tot}(t_2)$ in any other frame K', since $P_\mu^{tot}(t_1)$ and $P_\mu^{tot}(t_2)$ both transform the same way under Lorentz transf.

$$p_\mu^{tot}(t_1) = p_\mu^{tot}(t_2)$$

space components $\Rightarrow$ momentum conservation holds in all frames
time component $\Rightarrow$ energy conservation holds in all frames

(2) Why did we write Newton's eqn as $\frac{d\vec{p}}{dt} = \vec{F}$, with $\vec{p} = m\gamma\vec{v}$,

instead of $m\frac{d\vec{v}}{dt} = \vec{F}$ (as if used non-relativistic momentum)

If use $m\frac{d\vec{v}}{dt} = \vec{F}$, then $m\,\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\vec{F}$

$$\frac{1}{2}m\,d(v^2) = dt\,\vec{v}\cdot\vec{F} = d\vec{r}\cdot\vec{F} = dW$$

$$\Rightarrow \frac{1}{2}m\int d(v^2) = \int dW$$

$\frac{1}{2}mv^2 = W$ get non-relativistic kinetic energy

in this formulation, energy W is not the time component of any 4-vector. Therefore if energy was conserved in one frame K, it need not be conserved in another frame K'!

Only when we take $\frac{d\vec{p}}{dt} = \vec{F}$ with $\vec{p} = m\gamma\vec{v}$

do we get $\int \vec{F}\cdot d\vec{r} = m\gamma c^2 = p_0 c$ · time component of a 4-vector

$\Rightarrow$ energy conservation holds in all reference frames

Lorentz force in relativistic form

$$\frac{dp_\mu}{ds} = K_\mu$$

What is the K_μ that represents the Lorentz force? And how can we write it in a Lorentz covariant way?

K_μ should depend on the fields $F_{\mu\nu}$ and on the particle's trajectory x_μ

as $\vec{v} \to 0$ $\vec{K} = q\vec{E}$ (since magnetic force $\to 0$ as $\vec{v} \to 0$)

K_μ can't depend directly on x_μ as the force should be independent of where one puts the origin of the coordinates. So K_μ can depend only on derivatives $\dot{x}_\mu$, $\ddot{x}_\mu$, etc.

As $\vec{v} \to 0$, $\vec{K}$ does not depend on the acceleration, so $\vec{K}$ does not depend on $\ddot{x}_\mu$ or higher derivatives.

So K_μ depends only on $F_{\mu\nu}$ and $\dot{x}_\mu$

We need to form a 4-vector out of $F_{\mu\nu}$ and $\dot{x}_\mu$ that is <u>linear</u> in the fields $F_{\mu\nu}$ and proportional to the charge q. (since we want superposition to hold)

The only possibility is

$$K_\mu = q\, f(\dot{x}_\mu^2)\, F_{\mu\nu}\, \dot{x}_\nu$$

where $f(\mathring{x}_\mu^2)$ is some function of $\mathring{x}_\mu^2$.
But $\mathring{x}_\mu^2 = -c^2$ is a <u>constant</u>, so $f(\mathring{x}_\mu^2)$ is a constant.
That constant, $f(\mathring{x}_\mu^2) = 1$, is determined by the requirement that $\vec{K} = q\vec{E}$ as $\vec{v} \to 0$.

So we have $$\boxed{K_\mu = q F_{\mu\nu} \mathring{x}_\nu}$$

Let's check what this gives for the ordinary 3-force $\vec{F} = \frac{1}{\gamma}\vec{K}$

ith component $$F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma}\left(\sum_{j=1}^{3} F_{ij}\mathring{x}_j + F_{i4}\mathring{x}_4\right)$$

sub in for F_{ij} in terms of $\vec{A}$ $$= \frac{q}{\gamma}\left(\sum_{j=1}^{3}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\mathring{x}_j - \frac{iE_i}{c}(ic\gamma)\right)$$

use $\mathring{x}_4 = ic\gamma$

since $F_{i4} = \frac{-iE_i}{c}$

Now $$\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk} B_k$$

<u>proof:</u> $$\varepsilon_{ijk} B_k = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial A_m}{\partial x_l}$$ (using ε_{ijk} notation to take $\vec{\nabla}\times\vec{A}$)

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial A_m}{\partial x_l}$$

$$= \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$$

use $\mathring{x}_j = \gamma v_j$

So $$F_i = \frac{q}{\gamma}\sum_{j=1}^{3}\varepsilon_{ijk}B_k\gamma v_j + \frac{q}{\gamma}E_i\gamma$$

$$= q\sum_{j=1}^{3}\varepsilon_{ijk}B_k v_j + qE_i$$

$$= qE_i + q(\vec{v}\times\vec{B})_i$$

So $\boxed{\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}} = \frac{1}{\gamma}\vec{K}$

The Lorentz force has the same form in all inertial frames.
No relativistic modification is needed